Joint Matriculation Board

General Certificate of Education

Papers in

ADVANCED & SCHOLARSHIP MATHEMATICS

1960—1963

FURTHER MATHEMATICS

1961–1963

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JOHN SHERRATT & SON
Publishers : Altrincham

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Advanced and Scholarship

MATHEMATICS

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JOHN SHERRATT & SON
PUBLISHERS, ALTRINCHAM

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CONTENTS

June 1960

MATHEMATICS (Advanced)

Paper I

(Three hours)

Answer EIGHT questions.

1 sheet of graph paper supplied. Additional sheets will be supplied on request but all sheets issued must be placed within the answer-book and handed in to the Supervisor.

1. (*a*) The first term of a geometric series is 18 and the sum to infinity is 20. Find the common ratio and the sum of the first six terms. Find also in its simplest form the ratio of the nth term to the sum of all the subsequent terms of the infinite series.

(*b*) In the expansion in powers of x of the function

$$(1+x)(a-bx)^{12}$$

the coefficient of x^8 is zero. Find in its simplest form the value of the ratio a/b.

2. Write down the expansions of $\cos(A+B)$ and $\cos(A-B)$ in terms of cosines and sines of A and B.

(i) Find angles x and y, each between 0° and 90°, which satisfy the simultaneous equations

$$\cos x \cos y = 0.6,$$
$$\sin x \sin y = 0.2.$$

(ii) Prove that $\cos 3x = 4\cos^3 x - 3\cos x$. Hence find all the solutions, in the range $-180° < x \leqslant +180°$, of the equation

$$2\cos 3x + \cos 2x + 1 = 0.$$

3. A circle has centre O and radius r. Two parallel chords AB and CD are on the same side of O: the angle AOB is $\frac{1}{3}\pi$

radians and the angle COD is $(\frac{1}{3}\pi+2\theta)$ radians. Show that the area of the part of the circle between AB and CD is

$$\tfrac{1}{4}r^2\{4\theta+\sqrt{3}-2\sin(\tfrac{1}{3}\pi+2\theta)\}.$$

If θ is so small that θ^3 and higher powers may be neglected, deduce an approximation for this area in the form $a+b\theta+c\theta^2$, and state the values of the constants a, b, c.

4. Find the gradient of the tangent from the origin to the curve $y=\log_e x$. Hence, by considering a sketch of the curve, find the range of values of the constant k for which the equation $\log_e x=kx$ has two unequal roots.

Draw a graph of $y=\log_e x$ from $x=1$ to $x=1.9$, and use it to find to two decimal places the smaller root of the equation $4\log_e x=x$.

5. (*a*) Expand the function

$$\frac{e^{-2x}}{(1-x)^2}$$

in a series of ascending powers of x up to and including the term in x^3.

(*b*) Prove that

$$\frac{1+\cos\theta}{1-\cos\theta}=\cot^2\tfrac{1}{2}\theta.$$

Write down the first three terms in the expansion of $\log_e(1+x)$ in ascending powers of x. Express

$$\log_e\cot^2\tfrac{1}{2}\theta$$

as a series of powers of $\cos\theta$, giving the first three terms and the nth term.

6. A parallelogram $ABCD$ has its vertex A at the point $(1, 5)$; the order of the letters $ABCD$ indicates an anti-clockwise sense of rotation. The equation of the diagonal BD is $3x+4y=48$, and the length of BD is 30 units. Calculate the area of the parallelogram.

If C is the point (15, 7), calculate (i) the acute angle between AC and BD, (ii) the coordinates of the midpoint of AC.
Find the equation of AD.

7. Show that the equation of the normal at the point $P(at^2, 2at)$ on the parabola $y^2=4ax$ is

$$y+tx=2at+at^3.$$

This normal meets the x-axis at G; the mid-point of PG is M.

(i) Find the equation of the locus of M as P moves on the parabola

(ii) The focus of the parabola is S. Prove that SM is perpendicular to PG. If the triangle SPG is equilateral, calculate the coordinates of the possible positions of P.

8. (*a*) Differentiate

(i) $\dfrac{1+\tan x}{1-\tan x}$, (ii) $\sqrt{(1+\sin^2 x)}$, (iii) 2^x.

(*b*) If $y=e^{3x}\cos 4x$, find dy/dx and express it in the form $Re^{3x}\cos(4x+\alpha)$, where R is a positive constant; state the cosine and sine of the constant angle α.

Hence write down d^2y/dx^2 in a similar form.

9. (*a*) The pressure p units and the volume v units of an expanding gas are related by the law $pv^{1\cdot 4}=k$, where k is a constant. If the volume increases by 0.3 per cent, estimate the percentage change in the pressure.

(*b*) A right circular cone has base-radius a and height h. A circular cylinder is inscribed in the cone with one plane end on the base of the cone and the axis along the axis of the cone. If the height of the cylinder is x, show that the ratio of the volume of the cylinder to that of the cone is $3x(h-x)^2/h^3$.
Find the maximum value of this ratio as x varies.

10. Use the substitution $y = \sin x$ to evaluate

$$\int_0^{\alpha} \frac{\cos x \, dx}{2\cos 2x - 1},$$

where $\alpha = \sin^{-1}(\frac{1}{3})$.

(*b*) Evaluate

$$\int_0^1 \frac{\sin^{-1} x}{\sqrt{(1+x)}} \, dx.$$

June 1960

MATHEMATICS (ADVANCED)

PAPER II

(Three hours)

2 sheets of graph paper supplied for Alternative B. Additional sheets will be supplied on request but ALL sheets issued must be placed within the answer-book and handed in to the Supervisor.

Statistical tables provided for Alternative B.

Candidates must confine their attention to ONE only of the Alternatives, and that the Alternative for which they have been entered on the entry-form.

ALTERNATIVE A

Answer SEVEN questions

A1. A uniform rod PQ, of weight W and length $2a$, is smoothly pivoted at the end P. The rod rests in equilibrium with its mid-point G attached by a light elastic string of natural length a to a fixed point S at a height a vertically above P. In equilibrium the length of the string is $6a/5$. Show that the modulus of the string is $6W$.

A load of weight nW is now attached to a point L of the rod where $PL = xa$. If the new position of equilibrium of the rod is horizontal, find x in terms of n. Show that this equilibrium position is impossible unless $n \geqslant \frac{1}{2}(5 - 3\sqrt{2})$.

A2. Two uniform beams AB, AC, each of length $2a$ and of weights $4W$ and $3W$ respectively, are smoothly hinged at A and stand in equilibrium with the plane BAC vertical and the ends B and C on a rough horizontal plane. The angle BAC is 2θ.

(i) Calculate, in terms of W, the normal reactions at B and C.

(ii) Show that the forces of friction at B and C are each of magnitude $(7W \tan \theta)/4$.

If the coefficient of friction between each beam and the plane is μ, find the least value of μ in order that equilibrium can be preserved.

A3. Prove that the centre of mass of a uniform solid right cylinder cone, of height h and semi-vertical angle α is at a distance $\frac{3}{4}h$ from its vertex.

A frustum is cut from the cone by a plane parallel to the base at a distance $\frac{1}{2}h$ from the vertex. Show that the distance of the centre of mass of this frustum from its larger plane end is $11h/56$.

This frustum is placed with its curved surface in contact with a horizontal table. Show that equilibrium is not possible unless $45 \cos^2\alpha \geqslant 28$.

A4. A particle P is projected from a point O with velocity V at an angle of elevation θ. After time t the horizontal and upward vertical distances of P from O are x and y respectively. Show that the equation of the trajectory is

$$y = x \tan \theta - \frac{gx^2 \sec^2\theta}{2V^2}.$$

A ball is projected from the point O of a horizontal plane with speed 70 ft. per sec. and just passes over a wall at a point $12\frac{1}{2}$ ft. above the level of O and at a horizontal distance $87\frac{1}{2}$ ft. from O. Prove that there are two possible angles of projection one of which is $\tan^{-1}(\frac{1}{2})$; find the other. Find also the ratio of the maximum heights above the plane which the ball reaches in the two possible trajectories.

(Take g as 32 ft. per sec.^{2})

A5. One end of a light inextensible string is attached to a particle A of mass $2m$ which lies on a smooth horizontal table. The string passes over a small smooth pulley P at the edge of the table. To the other end of the string is attached a particle B of mass m. At time $t=0$, the system being at rest with the string taut and PB vertical, a variable force F is applied to A in the direction PA. If $F=mg(1+\cos\omega t)$ at time t and ω is a constant, show that in the subsequent motion the displacement x of A from its initial position in the direction PA satisfies the equation

$$d^2x/dt^2=\tfrac{1}{3}g\cos\omega t.$$

Find x in terms of g, ω and t. Show that the least value of the tension in the string is $\frac{2}{3}mg$.

Show that the maximum rate of working of the force F first occurs when $t=\frac{1}{3}\pi/\omega$.

A6. A cyclist working at $\frac{3}{25}$ horse power rides at a steady speed of 30 miles per hour down a slope of inclination $\sin^{-1}(1/80)$. If the total mass of the rider and his machine is 200 lb., show that the resistance opposing his motion is 4 lb. wt.

If the resistance varies directly as his velocity and his rate of working is unaltered, show that, when the cyclist is riding along a level road at speed v ft. per sec., his acceleration is $4(726-v^2)/(275v)$ ft. per sec.^{2}. Hence show that the interval of time during which his speed increases from $7\frac{1}{2}$ miles per hour to 15 miles per hour is

$$\frac{275}{8}\log_e\left(\frac{5}{2}\right)\text{ sec.}$$

(Take g as 32 ft. per sec.^{2})

A7. A particle moves with constant speed v in a circle of radius r. Show that the acceleration of the particle is v^2/r directed towards the centre of the circle.

A particle P of mass $2m$ is attached by a light inextensible string of length a to a fixed point O and is also attached by another light inextensible string of length a to a small ring Q of mass $3m$ which can slide on a fixed smooth vertical wire passing through O. The particle P describes a horizontal circle with OP inclined at an angle $\frac{1}{3}\pi$ with the downward vertical.

(i) Find the tensions in the strings OP and PQ.
(ii) Show that the speed of P is $(6ga)^{\frac{1}{2}}$.
(iii) Find the period of revolution of the system.

A8. Three particles A, B, C of masses m, $2m$, $3m$ respectively lie at rest in that order in a straight line on a smooth horizontal table. The distance between consecutive particles is a. A slack light inelastic string of length $2a$ connects A and B. An exactly similar slack string connects B and C. If A is projected in the direction CBA with speed V, find the time which elapses before C begins to move. Find also the speed with which C begins to move. Show that the ratio of the impulsive tensions in BC and AB when C is jerked into motion is $3:1$. Find the total loss of kinetic energy when C has started to move.

ALTERNATIVE B

Answer SIX questions

2 sheets of graph paper supplied. Additional sheets will be supplied on request but ALL sheets issued must be placed within the answer-book and handed in to the Supervisor.

Statistical tables are also provided.

B1. Prove that the number of permutations of n unlike things, taken r at a time, is $n!/(n-r)!$

Nine plots of land are to be used in an agricultural experiment and five different manurial treatments are to be given to five of the plots, the remaining four plots being left untreated. Calculate

(i) the number of ways in which the manurial treatments can be arranged,

(ii) the probability that a given plot will receive some sort of manurial treatment,

(iii) the probability that each of two given plots will receive specified manurial treatments.

B2. Samples of forty articles are selected at random from a large bulk of articles produced by a machine. The following list shows the number of defective articles in each of twenty such samples:

1, 0, 4, 0, 0, 4, 4, 1, 4, 2, 3, 3, 2, 3, 0, 2, 4, 2, 0, 1.

Calculate the mean number of defective articles per sample and estimate the percentage of defective articles in the total output of the machine.

Assuming that the binomial law applies,

(i) calculate the probability of there being *one or more* defective articles in a sample of forty,

(ii) find the least integer N such that the probability of there being N or more defective articles in a sample of forty is less than $\frac{1}{2}$.

Supposing that 10 per cent of the whole bulk is sampled in the above way and that the defective articles found in each sample are rejected, estimate the percentage of defective articles finally remaining in the bulk.

B3. A variate x can assume values only between 0 and 5 and the equation of its frequency curve is

$$y=A \sin \tfrac{1}{5}\pi x, \ (0 \leqslant x \leqslant 5),$$

where A is a constant such that the area under the curve is unity. Determine the value of A and obtain the median and quartiles of the distribution.

Show also that the variance of the distribution is

$$50\left\{\frac{1}{8}-\frac{1}{\pi^2}\right\}.$$

B4. Throughout Great Britain during 1958 the question "At about what age did you start to smoke a cigarette (or a pipe) a day for as long as a year?" was put to a random sample of 100 male smokers. The following table was drawn up from the answers received:

Age at which the man started to smoke	*Number of men*
15	27
16	19
17	12
18	16
19	5
20–24	15
25–29	4
30–34	1
35–39	1

Using 17 years as an arbitrary origin, or otherwise, calculate the mean age and the standard deviation of the ages at which the men began to smoke.

Given that the corresponding mean and the standard deviation for a random sample of 100 female smokers are 23.8 years and 9.8 years respectively, determine whether or not the difference between the means is significant.

B5. (*a*) In an examination the respective probabilities of three candidates solving a certain problem are $\frac{4}{5}$, $\frac{3}{4}$ and $\frac{2}{3}$. Calculate the probability that the examiner will receive from these candidates

(i) one, and only one, correct solution,

(ii) not more than one correct solution,

(iii) at least one correct solution.

(*b*) A bag contains 12 white balls and 8 black balls. Calculate the probability that a random sample of 5 balls drawn together from the bag will contain at least 4 white balls.

B6. An automatic machine produces bolts whose diameters are required to lie within the tolerance limits 0.496 in. to 0.504 in. A random sample of bolts produced by the machine is found to have a mean diameter of 0.498 in. and a standard deviation of 0.002 in. Assuming that the diameters are normally distributed, estimate the probability that any bolt produced by the machine will have a diameter outside the tolerance limits.

If the machine is adjusted to produce bolts of mean diameter 0.500 in., the standard deviation being unaltered, estimate the percentage of bolts likely to be rejected on full inspection.

B7. *American tourists visiting Great Britain.*

Year	*Year with 1953 as origin* t	*No. of tourists (thousands)* x	*Amount they spent (£ millions)* y
1953	0	185	36.1
1954	1	203	38.8
1955	2	240	42.5
1956	3	255	46.3
1957	4	266	47.7
1958	5	325	57.0
1959	6	360	64.0

Use the above data to plot graphs of (i) x against t and (ii) y against t and in each case draw by eye the straight line which best fits the points plotted. Obtain the equations of your straight lines in the form

$$x = m_1 t + c_1,$$

$$y = m_2 t + c_2.$$

Extend the table to show the estimated number of tourists and the amount which they will spend in each of the years 1960 and 1961. Discuss the reliability of these estimates.

ALTERNATIVE C

Answer SEVEN questions

C1 (*a*) Find the condition that must be satisfied by k in order that the expression

$$2x^2+6x+1+k(x^2+2)$$

may be positive for all real values of x.

(*b*) If a, b are two real numbers such that $a+b=1$, prove that $4ab\leqslant 1$.

Hence, or otherwise, show that $a^2+b^2\geqslant \frac{1}{2}$.

C2. (*a*) Given that the function

$$6x^4+x^3-25x^2-4x+4$$

has a factor of the form x^2-a^2, find a^2 and express the function as the product of four linear factors.

(*b*) Show that there is only one real root of the equation $x^3-3x-3=0$ and that its value lies between 2 and 3.
Find this value correct to one decimal place.

C3. Prove that

$$\begin{vmatrix} bc & b+c & a \\ ca & c+a & b \\ ab & a+b & c \end{vmatrix} =(b-c)(c-a)(a-b)(a+b+c).$$

Hence, or otherwise, determine the values of a for which the equations

$$\begin{aligned} 2x+3y \qquad &+a=0, \\ 2ax+(2+a)y+1&=0, \\ ax+(1+a)y+2&=0, \end{aligned}$$

are consistent.

C4. Write down the factors of 1001.

If $N=a_0+a_1t+a_2t^2+\ .\ .\ .\ +a_nt^n$

and $S=a_0-a_1+a_2\ \cdot\ .\ .\ .\ +(-1)^na_n$,

June 1961

MATHEMATICS (ADVANCED)

PAPER I

(Three hours)

Answer SEVEN questions

1. (*a*) Given the simultaneous equations

$$x^2 - 6xy + 11y^2 = 3a^2,$$
$$x^2 - 2xy - 3y^2 = 5a^2,$$

derive an equation in x and y only, and hence solve the equations for x and y in terms of a.

(*b*) Write down the nth term of the arithmetic series with first term a and common difference d.

In each of a set of n separate arithmetic series, the first term is 1. The common difference of the first series is 1, of the second 2, of the third 2^2, and so on. Find, in its simplest form, a formula for the sum of the nth terms of the n series.

2. (*a*) Find the ranges of values of k for which the equation

$$x^2 + (k-3)x + k = 0$$

has (i) real distinct roots, (ii) roots of the same sign.

(*b*) Solve the equation

$$x^2 + 4x + 20 = 0,$$

giving the roots in the form $p \pm iq$, where p and q are real.

(*c*) If $z^3 = 1$, find the two possible values of

$$1 + z + z^2.$$

3. (*a*) Given that

$$5 \cos \theta + 12 \sin \theta \equiv R \cos (\theta - \alpha),$$

where R and α are independent of θ and R is positive, obtain the values of R and α.

Hence find the values of θ between $-180°$ and $+180°$ which satisfy the equation

$$5 \cos \theta + 12 \sin \theta = 3.25,$$

giving the answers to the nearest minute.

(*b*) Given that

$$\sin 2\alpha + \sin 2\beta = p,$$
$$\cos 2\alpha + \cos 2\beta = q,$$

prove that

$$\frac{p}{q} = \tan (\alpha + \beta).$$

Prove also that

$$\frac{4p}{p^2 + q^2 + 2q} = \frac{\sin (\alpha + \beta)}{\cos \alpha \cos \beta},$$

and deduce an expression for $\tan \alpha \tan \beta$ in terms of p and q.

4. (*a*) Write down the first four terms in the expansion in ascending powers of x of $(1 + 4x)^{\frac{1}{2}}$, and simplify the coefficients.

Hence, by putting $x = -1/100$, calculate $\sqrt{6}$ correct to four decimal places.

(*b*) Write down, as far as the terms in x^4 inclusive, the expansions in ascending powers of x of (i) $\log_e (1 + x)$, (ii) e^x.

If $|x| < 1$ and

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \ldots,$$

find an expression, in a form not involving an infinite series, for y in terms of x. Hence find the expansion of x in powers of y as far as the term in y^4.

5. Show that the equation of the tangent to the curve $x = 3t^2$, $y = 2t^3$ at the point $P(3p^2, 2p^3)$ is

$$px - y = p^3.$$

If Q is the point $(3q^2, 2q^3)$, find the coordinates of the point of intersection T of the tangents at P and Q.

If the tangents at P and Q make angles θ and $\frac{1}{2}\pi - \theta$ respectively with the x-axis, find the relation between p and q. Hence find the (x, y) equation of a curve on which T lies for all values of θ.

Show in a sketch the given curve. Show in the same sketch the curve on which T lies, and indicate the part of this curve which is the locus of T as θ varies.

6. If $P(x_1, y_1)$ is a point outside the circle

$$x^2+y^2+2gx+2fy+c=0,$$

show that the length of the tangent PT from P to the circle is given by

$$PT^2=x_1^2+y_1^2+2gx_1+2fy_1+c.$$

Two circles have centres $A(1, 3)$ and $B(6, 8)$, and intersect at $C(2, 6)$ and D. Find the equation of each of the circles and that of the line CD.

The tangents to the circles from a point P are of equal length. Verify that P lies on the line CD.

7. A particle moves along a straight line Ox in the time interval $0 \leqslant t \leqslant \pi$; after t seconds its displacement from O is x feet where

$$x=t+\sin 2t.$$

Calculate the values of t between 0 and π when the direction of motion changes, and show that the particle always remains on the same side of O. Find also the time at which the acceleration is zero.

Sketch a graph of x for $0 \leqslant t \leqslant \pi$, and state the largest value of x in this interval.

Calculate the mean value of x with respect to time for $0 \leqslant t \leqslant \pi$.

8. (*a*) Differentiate

(i) $\sin^{-1} 2x$, (ii) x^2e^{-x}.

(*b*) Express in partial fractions the function

$$\frac{8x+15}{(x^2+4)(x-3)},$$

and hence integrate this function.

9. (*a*) Find by integration by parts the area bounded by the curve.

$$y=\frac{\log_e x}{\sqrt{x}},$$

the x-axis between $x=1$ and $x=e$, and the line $x=e$. Give the answer correct to three decimal places.

(*b*) If $y=x+\frac{1}{x}$, calculate $\frac{dy}{dx}$ and sketch the curve for $x>0$.

The area between this curve, the x-axis, and the lines $x=1$, $x=3$ is turned through a complete revolution about the x-axis. Calculate, as a multiple of π, the volume of the solid generated.

MATHEMATICS (ADVANCED)

PAPER II

(Three hours)

Answer SEVEN questions.

1. Write down the general solution of the differential equation

$$\frac{dy}{dt}=-ky,$$

where k is a constant.

A radioactive substance disintegrates at a rate proportional to its mass. If the mass remaining at time t is m, show that

$$m=m_0e^{-kt},$$

where m_o is the initial mass and k is a constant.

One-third of the original mass of the substance disintegrates in 70 days. Calculate, correct to the nearest day, the time required for the substance to be reduced to half its original mass. If the original mass was 100 gm., calculate, correct to the nearest gm., the mass remaining after 210 days.

2. A non-uniform rigid beam AB, of length $3a$ and weight nW, rests on supports P and Q at the same level, where $AP=PQ=QB=a$. When a load of weight W is hung from A, the beam is on the point of tilting about P. Find the distance of the centre of gravity of the beam from A. When an additional load of weight W_1 is hung from B, the forces exerted on the supports at P and Q are equal. Find W_1 in terms of n and W.

If a couple, of moment L and acting in the vertical plane through AB, is now applied to the loaded beam, the reaction at P is increased in the ratio 3:2. Show that

$$L=\tfrac{1}{3}(n+1)Wa.$$

3. Four equal uniform rods, each of weight W, are smoothly jointed together at their ends to form a rhombus $ABCD$. The rhombus is suspended from A and is maintained in equilibrium, with C below A and with $\angle DAB=2\theta$, by a light string connecting the joints at A and C. Find the horizontal and vertical components of the force exerted by AB on BC. Hence, or otherwise, find the tension in the string.

4. Two small rings A, B, each of weight W, are threaded on a fixed rough horizontal wire. The rings are connected by a light inelastic string, of length $2a$, to the mid-point C of which is attached a particle of weight $2W$. The system rests in equilibrium with $\angle ACB=2\beta$.

(i) Find the tension in each part of the string.

(ii) Find, in terms of W and β, the normal reactions and the frictional forces between the wire and the rings. (The directions of the frictional forces acting on the rings must be clearly shown in a diagram.)

If the coefficient of friction between each ring and the wire is $\tfrac{3}{8}$, show that $AB\leqslant 6a/5$.

5. State Newton's laws of motion.

A particle P of mass m starts from rest at the point O and moves along a straight line so that when $OP=x$ the force acting on the particle is along the line and of amount $F(x)$. Use Newton's laws of motion to show that the gain of kinetic energy of the particle after it has moved a distance s is

$$\int_0^s F(x)dx.$$

A body of mass 144 lb. is pulled along a smooth horizontal plane by a horizontal force which is constant in direction and which diminishes uniformly with the distance the body has moved. The force is initially 4 lb. wt. and falls to 2 lb. wt. when the body has moved 25 ft. If the body starts from rest, find its speed when it has moved 36 ft.

(Take g as 32 ft. per sec.^{2})

6. A smooth narrow tube is in the form of a circle, of centre O and radius a, fixed with its plane vertical. Two particles A and B, of masses m and $2m$ respectively, are connected by a light inextensible string of length πa and are placed at opposite ends of the horizontal diameter of the tube, the string occupying the upper half of the tube. The system is released from rest. If OB makes an angle θ with the horizontal at time t after release and $0<\theta<\frac{1}{2}\pi$, show by energy considerations that

$$a\left(\frac{d\theta}{dt}\right)^2=\frac{2}{3}g\sin\theta.$$

Hence, or otherwise, find in terms of m, g and θ the tension in the string. Find also the force exerted by the tube on particle A.

7. Using a clearly marked diagram, state Newton's experimental law for the direct impact of two spheres.

A sphere moving on a smooth horizontal plane strikes a fixed smooth vertical plane with velocity V at an angle α to the plane. The coefficient of restitution between the sphere and the plane is e. Immediately after impact the velocity makes an angle β with the plane. Show that

$$\tan \beta = e \tan \alpha.$$

The sphere then strikes a second vertical plane which is perpendicular to the first vertical plane, the coefficient of restitution again being e. Prove that after the second impact the sphere will be moving with speed eV along a line parallel to the original direction of motion.

8. Write down the general solution of the differential equation

$$\frac{d^2y}{dt^2} = -n^2y,$$

where n is a constant, and find the particular solution for which $y=a$ and $\frac{dy}{dt}=0$ when $t=0$.

A rough horizontal board is made to move horizontally and performs simple harmonic oscillations of amplitude 2 ft. and period 6 sec. A particle is placed on the board and does not move relative to it. Giving your answers correct to two significant figures, find

(i) the greatest speed, in ft. per sec., of the particle,
(ii) the least possible value of the coefficient of friction between the particle and the board.

(Take g as 32 ft. per sec.^{2} and π as 3.14.)

9. An engine of mass 150 tons starts from rest at the top of a slope, of length 810 ft. and inclination α to the horizontal, where $\sin \pi = \frac{3}{280}$, and moves with its power and brakes switched off. The resistance opposing the motion of the engine is 10 lb. wt. per ton. At the bottom of the slope the engine moves on to a level track without change of speed and at once collides with a truck of mass 30 tons moving with speed 12 ft. per sec. in the same direction. After the collision the engine and truck move on together along the level track. Find their common velocity immediately after the collision.

If the engine is started immediately after the collision and works at the rate of 264 horse power, find the acceleration, in ft. per sec.^{2}, of the train when its speed is v miles per hour, assuming that the resistance to motion of the truck is 10 lb. wt. per ton.

(Take g as 32 ft. per sec.^{2})

June 1961

FURTHER MATHEMATICS (ADVANCED)

PAPER I

(Three hours)

Answer SEVEN questions.

1. (*a*) Prove that the sum of the squares of the first n positive integers is $n(n+1)(2n+1)/6$.

(*b*) Express $\dfrac{2-r}{r(r+1)(r+2)}$ in partial fractions.

Find the sum of the first n terms of the series whose rth term is $\dfrac{2-r}{r(r+1)(r+2)}$.

2. (*a*) Defining ${}_nC_r$ as the number of combinations of n things taken r at a time, and without using its value, prove that

$${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}.$$

(*b*) Find how many odd numbers greater than 400 can be formed from some or all of the digits 1, 2, 3, 4, 5, no repetitions being allowed.

3. If each of the elements of one row of a determinant of the third order is multiplied by the same number and added to the corresponding element of another row, prove that the value of the determinant is unaltered.

Without expanding the determinant, prove that $(a+b)$ is a factor of

$$\begin{vmatrix} 2a-b & b+c & a+2c \\ a-2b & c-a & 2c-b \\ b+c & b-2c & 2a-b \end{vmatrix}$$

Express the value of the determinant as a product of three factors, each linear in a, b, c.

4. Sketch the curve whose parametric equations are

$$x=a(\theta-\sin\theta), \qquad y=a(1-\cos\theta)$$

from $\theta=0$ to $\theta=2\pi$.

If s is the length of the curve from the origin to the point of parameter θ, prove that $\dfrac{ds}{d\theta}=2a\sin\tfrac{1}{2}\theta$. Find the length of the curve from $\theta=0$ to $\theta=2\pi$.

The curve is rotated about the x-axis to form a surface of revolution. Prove that the area of this surface is $64\pi a^2/3$.

5. (*a*) If l, m, n are the direction cosines of a straight line, prove that

$$l^2+m^2+n^2=1.$$

(*b*) A plane P is perpendicular to each of the planes $x+y+3z=0$, $3x-2y+4z=0$, and passes through the point $(1, 1, 1)$. Find

(i) the direction ratios of the normal to P,
(ii) the equation of P,
(iii) the coordinates of the foot of the perpendicular to P from the origin.

6. Show that a set of coplaner forces is equivalent to a single force acting at an arbitrary point of their plane together with a couple. Hence prove that the set of forces is equivalent to either a single force or a couple.

Forces whose components are

$$(P, 2P),\ (-P, P) \text{ and } (4P, 0)$$

act respectively at the points whose coordinates (x, y) are

$$(a, 0), (a, -a) \text{ and } (0, a).$$

Reduce the system to a force at the origin and a couple, and deduce that the resultant is a force acting in the line

$$4y-3x=2a.$$

7. A particle is projected under gravity with velocity V in a direction inclined to the horizontal at an angle θ. Derive expressions for the horizontal and vertical displacements, x and y, at time t after projection, and deduce an equation for the path of the particle.

A vertical section of a valley is in the form of a parabola $x^2=4ay$, where a is a positive constant and the axis of y is vertically upwards. A gun placed at the origin fires a shell with velocity $\sqrt{(2gh)}$ at an angle θ to the horizontal. If the shell strikes the section at the point (x, y), prove that

$$x=\frac{4ah\tan\theta}{a+h+a\tan^2\theta}.$$

Deduce the greatest value of x as θ varies.

8. Two sets of particles in the same plane have total masses M_1, M_2 and centres of mass G_1, G_2. Show that the centre of mass of the combined sets is the same as that of particles of masses M_1, M_2 situated at G_1, G_2.

The base of a uniform solid hemisphere of radius r and density σ is fitted to the base of a uniform solid right-circular cone, radius r, height h and density 3σ. The composite solid rests with its hemispherical end on a fixed horizontal plane. Show that if $h<r$ the solid may be tilted through any acute angle from the upright position without causing it to topple over.

9. A particle of mass m moves in a straight line and x is its displacement, at time t, from a fixed point of the line.

Explain the nature of the forces acting on the particle, given that the equation of motion is

$$\frac{d^2x}{dt^2}+2k\frac{dx}{dt}+5k^2x=0,$$

where k is a positive constant.

Solve the equation of motion given that $x=0$, $dx/dt=u$ when $t=0$.

Show that, when x is next zero,

$$\frac{dx}{dt}=-ue^{-\pi/2},$$

and find the corresponding value of d^2x/dt^2.

FURTHER MATHEMATICS (ADVANCED)

PAPER II

(Three hours)

Answer SEVEN questions

1. (*a*) The product of two of the roots of the equation

$$x^4+4x^3+kx^2-32x-21=0$$

is -3. Find (i) the value of k, (ii) the four roots of the equation.

(*b*) For each root of the equation

$$x^3-3x+1=0$$

obtain a pair of consecutive integers c and $c+1$ between which the root is situated.

2. If n is a positive integer, prove that

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta.$$

If $2\cos\theta=x+x^{-1}$, express x in terms of θ and deduce that $2\cos n\theta=x^n+x^{-n}$. Hence, or otherwise, express $\cos^5\theta$ as a sum of cosines of integral multiples of θ.

3. Prove that, for a given real number u, there is exactly one real number x such that $\sinh x=u$, and that

$$x=\log_e[u+\sqrt{(u^2+1)}].$$

Find how many real numbers y satisfy $\cosh y = v$, if $v > 1$.

If $\sinh x = \frac{3}{4}$, $\cosh y = \frac{13}{12}$, show that

$$x + y = \log_e 3 \text{ or } \log_e(4/3).$$

4. (*a*) Sketch the curve whose equation in polar coordinates is

$$r = \frac{3}{1 + \cos\theta},$$

for $0 \leqslant \theta \leqslant 2\pi$.

If ϕ is the angle between the radius vector and the tangent at the point (r, θ) on this curve, prove that

$$2\phi + \theta = \pi.$$

(*b*) Obtain the value of $f(a)$, where

$$f(a) = \int_0^a e^{-2x} \sin x \, dx, \quad (a > 0).$$

Prove that $f(a)$ tends to the limit $\frac{1}{5}$ when a tends to infinity.

5. The only force acting on a particle moving in a straight line is a resistance $mk(c^2 + v^2)$ acting in that line; m is the mass of the particle, v its velocity and k, c are positive constants. The particle starts to move with velocity U and comes to rest in a distance s; its speed is $\frac{1}{3}U$ when it has moved a distance $\frac{1}{2}s$. Show that $63c^2 = U^2$. Show also that when the distance moved is x

$$63 \frac{v^2}{U^2} = 64e^{-2kx} - 1.$$

6. A light spring AB, of modulus nmg and unstretched length l, has a particle of mass m attached to each end, and lies at rest on a fixed smooth horizontal plane. If the particle at B is now projected with velocity U in the direction AB, show, by using the principles of energy and momentum or otherwise, that when the spring has a length $l + x$ and x is increasing, the velocity of A is

$$\tfrac{1}{2}U - \tfrac{1}{2}\sqrt{\left\{ U^2 - \frac{2ngx^2}{l} \right\}},$$

and find the velocity of B.

Hence find the greatest value of x during the motion.

7. Two equal uniform circular cylinders B and C, each of weight W, rest side by side on a rough horizontal plane; their axes are horizontal and parallel and the cylinders are almost in contact. A third equal cylinder A is gently placed so as to rest in equilibrium upon them and in contact with each of them along a generator. Show, by considering the equilibrium of C, that the force of friction between C and A and that between C and the plane are equal in magnitude. Show also that their common magnitude is $\frac{1}{2}W(2-\sqrt{3})$.

If μ is the coefficient of friction at all contacts, show that $\mu > 2-\sqrt{3}$.

8. The motion of a particle P, whose coordinates are (x, y) referred to a pair of fixed axes through a point O, satisfies the equations

$$\frac{d^2x}{dt^2} = -\omega^2 x, \qquad \frac{d^2y}{dt^2} = -\omega^2 y;$$

the initial conditions are

$$x=a,\ y=0,\ \frac{dx}{dt}=0 \text{ and } \frac{dy}{dt}=b\omega \text{ when } t=0,$$

where ω, a and b are positive constants. Prove that the path of the particle is the ellipse $(x/a)^2+(y/b)^2=1$.

Find, in terms of OP, the magnitude of the component perpendicular to OP of the velocity of P. Hence, or otherwise, show that the angular speed of OP is

$$\frac{\omega ab}{OP^2}.$$

[The general solutions of the equations

$$\frac{d^2x}{dt^2} = -\omega^2 x, \qquad \frac{d^2y}{dt^2} = -\omega^2 y$$

may be quoted.]

9. Show that the moment of inertia of a uniform rod, of mass m and length l, about a perpendicular axis through its centre of mass is $ml^2/12$.

A uniform rod AB, of mass m and length $4a$, is smoothly pivoted at a point O of its length, where $AO=a$, and hangs at rest with A uppermost. The rod receives a horizontal impulse of magnitude J at its centre of mass. Find the initial angular velocity of the rod.

If the rod completes revolutions in the subsequent motion, find an inequality for J in terms of a, m and g.

June 1962

MATHEMATICS (ADVANCED)

PAPER I

(Three hours)

Negligently presented or slovenly work will be penalized

Answer SEVEN questions

1. (*a*) Given that α is a root of the equation

$$x^2-5x+3=0,$$

state the values of $\alpha^2-5\alpha$ and $\alpha^3-5\alpha^2+3\alpha$. Hence show that $\alpha^3=22\alpha-15$. If β is the other root of the equation, find without solving the equation the value of $\alpha^3+\beta^3$, and form the quadratic equation whose roots are α^2/β and β^2/α.

(*b*) Find the real and imaginary parts of

$$\frac{2+3i}{3+4i}.$$

(*c*) If the complex number $4+7i$ is represented by the point P on the Argand diagram, write down the complex numbers which are represented by

(i) the reflection of P in the x-axis,

(ii) the reflection of P in the line $y=x$,

(iii) the reflection of P in the line $y=-x$.

2. (*a*) If θ is not a multiple of $\pi/2$, and if x, y, z are given as sums of the following infinite geometric series

$$x=1+\cos^2\theta+\cos^4\theta+\ldots,$$
$$y=1+\sin^2\theta+\sin^4\theta+\ldots,$$
$$z=1+\cos^2\theta\sin^2\theta+\cos^4\theta\sin^4\theta+\ldots,$$

prove that

(i) $x+y=xy$,

(ii) $x+y+z=xyz$.

(*b*) Express the function

$$\frac{18x-10x^2}{(3-x)(1-x)^2}$$

as a sum of partial fractions, and hence expand this function in ascending powers of x as far as the term in x^3.

3. (*a*) Find in the range $-180° < x < 180°$ the solutions of the equation $\cos 5x=\cos x$.

(*b*) Prove that

$$\frac{1+\cos\theta+\sin\theta}{1-\cos\theta+\sin\theta}=\frac{1+\cos\theta}{\sin\theta}.$$

(*c*) In a triangle ABC, the angle $C=2\pi/3$. Express c^2 in terms of a and b.

The triangle is rotated about A, in its own plane, through an angle $\theta(<\pi)$. Find, in terms of a, b and θ, the area swept out (i) by AC, (ii) by BC.

4. Sketch the graph of

$$y=\frac{x}{1+x}.$$

From your graph, or otherwise, find the range of values of x for which $|y|<1$.

Express $1+x$ in terms of y, and hence expand $\log_e(1+x)$ in ascending powers of y, giving the first three terms and the nth term and stating the range of values of x for which the expansion is valid.

Hence or otherwise show that when $x>0$

$$\log_e(1+x)>\frac{x}{1+x}.$$

5. (*a*) A circle with centre (3, 2) touches the line $4x-3y+4=0$. Find the equation of the circle and show that the circle touches the x-axis.

(*b*) In each of the following pairs of equations, t is a parameter. Sketch the locus given by each pair of equations.

(i) $x=3+5\cos t,\ \ y=4+5\sin t \qquad (0 \leqslant t \leqslant \pi)$,

(ii) $x=3\cos t,\ \ y=4\cos t \qquad (0 \leqslant t \leqslant \pi)$,

(iii) $x=3+t\cos\frac{\pi}{3},\ y=4+t\sin\frac{\pi}{3} \qquad (-\infty < t < \infty)$.

6. The points $P(ap^2, 2ap)$ and $Q(aq^2, 2aq)$ move on the parabola $y^2=4ax$, and $p+q=2$. Show that the chord PQ makes a constant angle with the x-axis, and that the locus of the mid-point M of PQ is part of a line which is parallel to the x-axis.

If also the point $R(ar^2, 2ar)$ moves so that $p-r=2$, find in its simplest form the (x, y) equation of the locus of the mid-point N of PR.

7. In the triangle ABC, the sides AB, AC are equal and contain an angle 2θ. The circumscribed circle of the triangle has radius R. Show that the sum of the lengths of the perpendiculars from A, B, C to the opposite sides of the triangle is

$$2R(1+4\sin\theta-\sin^2\theta-4\sin^3\theta).$$

If R is constant and θ varies, show that this sum has only one stationary value, and that this value is a maximum.

8. (*a*) Show that the second differential coefficient with respect to x of $\log_e(1+\sin x)$ is never positive.

(*b*) Prove that the gradient of the curve

$$y=x^3+6x^2+15x+36$$

is positive for all values of x. Show that the curve has a point of inflexion when $x=-2$, and state the gradient of the curve at this point.

Write down the equation of the tangent to the curve at the point where $x=0$, and find the coordinates of the point where this tangent meets the curve again.

9. (*a*) Evaluate

$$\text{(i)} \int_0^{\frac{1}{4}\pi} \sin 5x \cos 3x \, dx,$$

$$\text{(ii)} \int_0^{1} xe^{-3x} \, dx.$$

(*b*) Show that

$$\int_0^{\frac{1}{4}\pi} (\tan^3 x + \tan x) \, dx = \tfrac{1}{2},$$

and hence or otherwise evaluate

$$\int_0^{\frac{1}{4}\pi} \tan^3 x \, dx.$$

MATHEMATICS (ADVANCED)

PAPER II

(Three hours)

Negligently presented or slovenly work will be penalized.

Answer SEVEN questions.

1. A rigid body is in equilibrium under the action of three coplanar forces. Show that the lines of action of the forces are either concurrent or parallel.

A uniform smooth sphere, of weight W and radius a, rests on a smooth plane which is inclined at an angle β to the horizontal. The sphere is held in equilibrium by a light string of length $8a/5$ joining a point of its surface to a point of the plane. Find, in terms of W and β, the tension of the string and the force exerted by the plane on the sphere.

2. Prove that the centre of mass of a uniform solid hemisphere of radius a is at a distance $3a/8$ from the centre of the plane base of the hemisphere.

The plane base of the hemisphere coincides with the base of a uniform solid right circular cone, of base-radius a and height a. The two solids are made out of the same material and together form one uniform composite solid. Show that the centre of mass of this composite solid is at a distance $7a/6$ from the vertex of the cone.

This solid, of weight W, rests with a point of the curved surface of the hemisphere on a horizontal plane and with its axis inclined at an angle θ to the horizontal, equilibrium being maintained by a couple acting in a vertical plane. Calculate the moment of this couple and indicate its sense in a diagram.

3. Three uniform rods AB, BC, CD, each of length $2a$ and of weights W, $4W$, W respectively, are smoothly jointed at B and C. The rods are in equilibrium in a vertical plane with A and D resting on rough horizontal ground and AB, CD each inclined at 60° to the horizontal so that $AD=4a$.

(i) Calculate the vertical components and the frictional components of the forces at A and D.

(ii) Calculate the horizontal and vertical components of the force exerted by BC on AB.

(iii) If μ is the coefficient of friction at A and D, find the inequality satisfied by μ.

(The directions of the forces acting on each rod must be clearly indicated in a diagram.)

4. Two particles A, B are simultaneously projected from a point O, and they afterwards move under gravity in the same vertical plane. Show that the velocity of B relative to A is constant. Hence, or otherwise, show that the distance AB is proportional to the time.

The speeds of projection of A and B are each 80 ft. per sec. and the particles both strike the horizontal plane through O at a point N, where $ON = 192$ ft. If A reaches N sooner than B, verify that the angle of projection of A is $\sin^{-1}$ $(\frac{3}{5})$. Find the distance AB at the instant when A reaches N.

(Take g as 32 ft. per sec.^{2})

5. One end of a light inextensible string is attached to a particle A of mass $2m$ which lies on a smooth horizontal table. The string passes over a small smooth fixed pulley P at the edge of the table and hangs in a loop on which a smooth ring B of mass $8m$ is threaded. The string passes up and over a second smooth fixed pulley Q and is finally attached to a particle C of mass m. The system is released from rest at time $t=0$ with the string taut and with the portions PB, BQ, QC vertical. If f_1 and f_2 are respectively the accelerations of A and C in the directions AP and CQ, show that the acceleration of B is $\frac{1}{2}(f_1+f_2)$ downwards. Show that the tension in the string is $3mg/2$, and determine f_1 and f_2 in terms of g.

Calculate, in terms of m, g and t, the gain in kinetic energy of the system at time t and verify that this gain of energy is equal to the loss of potential energy of the system.

6. A lorry of mass M tons moves along a straight level road with its engine shut off. When the lorry is travelling at v ft. per sec. the resistance to its motion is $a(v^2+V^2)/V^2$ lb. wt., where a and V are positive constants. Find, in ft. per sec.^{2}, the retardation of the lorry. Prove that the distance in which the lorry is brought to rest from a speed of V ft. per sec. is

$$\frac{35MV^2\log_e 2}{a}\text{ft.}$$

and that the corresponding time is

$$\frac{35\pi MV}{2a}\text{sec.}$$

(Take g as 32 ft. per sec.^{2})

7. A particle moves with constant speed v in a circle of radius r. Show that the acceleration of the particle is v^2/r directed towards the centre of the circle.

A rough horizontal plate rotates with constant angular velocity ω about a fixed vertical axis. A particle of mass m lies on the plate at a distance $5a/4$ from this axis. If the coefficient of friction between the plate and the particle is $\frac{1}{3}$ and the particle remains at rest relative to the plate, show that

$$\omega \leqslant \sqrt{\left(\frac{4g}{15a}\right)}.$$

The particle is now connected to the axis by a horizontal light elastic string, of natural length a and modulus $3mg$. If the particle remains at rest relative to the plate and at a distance $5a/4$ from the axis, show that the greatest possible angular velocity of the plate is

$$\sqrt{\left(\frac{13g}{15a}\right)}$$

and find the least possible angular velocity.

8. State the law of conservation of linear momentum for two interacting particles. Show how the law of conservation of linear momentum applied to two particles which collide directly follows from Newton's laws of motion.

Three smooth spheres A, B, C, equal in all respects, lie at rest and separated from one another on a smooth horizontal table in the order A, B, C with their centres in a straight line. Sphere A is projected with speed V directly towards sphere B. If the coefficient of restitution at each collision is e, where $0<e<1$, find the velocity of each of the spheres just after C is set in motion. Show that A strikes B a second time.

9. A particle P of mass m moves along a straight line so that, at time t, its displacement $OP(=x)$ from a fixed point O in the line is given by

$$x=a\cos(\omega t+\varepsilon),$$

where a, ω, ε are constants. Find, in terms of a, x and ω, (i) the speed v, (ii) the acceleration f of P. Find also, in terms of m, x and ω the force acting on P in the direction of x increasing.

The ends of a light elastic string, of natural length $2l$ and modulus λ, are attached to fixed points A, B of a smooth horizontal table, where $AB=3l$. To the mid-point of the string is attached a particle of mass m. The particle is displaced from its equilibrium position through a distance $\frac{1}{4}l$ towards B and is released from rest. Show that the subsequent motion of the particle is simple harmonic and find its period. Find also the greatest speed and the greatest acceleration of the particle.

June 1962

FURTHER MATHEMATICS (ADVANCED)

PAPER I

(Three hours)

Negligently presented or slovenly work will be penalized.

Answer SEVEN questions.

1. (*a*) Find the general solution of the equation

$$\tan 2x = 2 \sin x.$$

(*b*) The points A, B, C, D, in the Argand diagram correspond to the complex numbers a, b, c, d, respectively. Prove that

(i) If $a-b+c-d=0$, then $ABCD$ is a parallelogram,

(ii) If also $a+ib-c-id=0$, then $ABCD$ is a square.

2. (*a*) If all the roots of the equation

$$x^n + p_1 x^{n-1} + \ldots + p_n = 0$$

are real, prove that $p_1^2 - 2p_2 \geqslant 0$.

(*b*) An equation in y is formed from the cubic equation

$$a_0 x^3 + a_1 x^2 + a_2 x + a_3 = 0$$

by substituting $y+k$ for x. Find the value of k for which the coefficient of y^2 in the new equation is zero.

Solve the equation

$$4x^3 - 12x^2 + 13x - 6 = 0.$$

3. The points A, B have coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) respectively. The point E on AB is such that $AE/EB = l/m$. Prove that the x-coordinate of E is $(mx_1 + lx_2)/(l+m)$.

The vertices of a triangle are $A(3, 5, 2)$, $B(2, 3, 4)$, $C(5, 2, -4)$. The internal bisector of the angle A meets BC at D. Find (i) the coordinates of D and (ii) the equation of the plane through D perpendicular to BC.

4. Prove that

$$\int_0^a f(x)\,dx = \int_0^{\frac{1}{2}a} [f(x) + f(a-x)]\,dx.$$

Use this result to show that

(i) $\displaystyle\int_0^{\pi} x \sin^n x\,dx = \pi \int_0^{\frac{1}{2}\pi} \sin^n x\,dx,$

(ii) $\displaystyle\int_0^{\pi} \frac{x \sin x\,dx}{\sqrt{(1 + \tan^2\alpha \sin^2 x)}} = \frac{\pi\alpha}{\tan\alpha}$ if $0 < \alpha < \frac{1}{2}\pi$.

5. The parametric equations of a curve (the tractrix) are

$$x = \frac{c}{2}\log\frac{1+\sin\theta}{1-\sin\theta} - c\sin\theta,\quad y = c\cos\theta,$$

for $0 \leqslant \theta < \frac{1}{2}\pi$. The length of the curve is s, measured from the point A where $\theta = 0$ to the point P whose parameter is θ. Prove that $\dfrac{ds}{d\theta}$ $c\tan\theta$ and deduce that $s = c\log(c/y)$.

The arc AP is rotated about the x-axis through an angle 2π. Prove that the area of the curved surface so formed is $2\pi c(c-y)$.

6. A heavy uniform circular hoop hangs on a small rough horizontal peg. The hoop is pulled by a gradually increasing horizontal force which acts in the plane of the hoop and is applied at the other end of the diameter through the peg. If the hoop has not slipped when this diameter makes an acute

angle θ with the vertical, show that the ratio of the frictional force to the normal reaction at the peg is

$$\tan\theta/(2+\tan^2\theta).$$

Find the least value of the coefficient of friction at the peg which will ensure that the hoop will not slip however hard it is pulled.

7. A particle P of mass m is suspended from a fixed point by an elastic string, and the equilibrium extension of the string is b. Show that the period of vertical oscillations of P is $2\pi\sqrt{(b/g)}$, assuming that the string remains in tension during the motion.

A second particle Q of mass $2m$ is now attached to P. Find the new position of equilibrium. If Q falls off, show that P will ascend for a time.

$$\left(\frac{2}{3}\pi+\sqrt{3}\right)\sqrt{\frac{b}{g}}$$

before descending again.

(Assume that P does not reach the fixed end of the string.)

8. A train of mass M moves on a straight horizontal track. At speeds less than V the resultant force on the train is constant and equal to P; at speeds not less than V the rate of working of the force is constant and equal to PV. Show that a speed v ($>V$) is attained from rest in the time

$$\frac{M(V^2+v^2)}{2PV},$$

and find the corresponding distance travelled.

9. Show that radius of curvature of the parabola $x^2=4ay$ at the point $(2ap, ap^2)$ is $2a(1+p^2)^{3/2}$.

A small heavy bead can slide freely along a fixed smooth wire in the form of this parabola, the axis of y being directed vertically upwards. The bead is projected from the origin with speed U. Show that when the direction of motion has turned through an angle ψ the speed of the bead is

$$\sqrt{(U^2 - 2ga \tan^2\psi)}$$

and that the force exerted by the bead on the wire is proportional to $\cos^3\psi$.

(Any formula connected with curvature may be quoted without proof.)

FURTHER MATHEMATICS (ADVANCED)

PAPER II

(Three hours)

Negligently presented or slovenly work will be penalized.

Answer SEVEN questions.

1. (*a*) Solve each of the differential equations

$$\text{(i)} \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 0,$$

$$\text{(ii)} \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 5x = 0.$$

(*b*) A particle moves in a straight line so that its distance x from a fixed point in the line satisfies the differential equation

$$\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 4x = 0.$$

The particle starts from rest at time $t=0$ when $x=a$. Prove that its greatest speed in the ensuing motion is $2ae^{-1}$.

2. (*a*) Prove that

$$\begin{vmatrix} a & b & c \\ a & a+b & a+b+c \\ a & 2a+b & 3a+2b+c \end{vmatrix} = a^3.$$

(*b*) Express

$$\begin{vmatrix} (y-z)^2 & (z-x)^2 & (x-y)^2 \\ yz & zx & xy \\ 1 & 1 & 1 \end{vmatrix}$$

as a product of factors which are linear in x, y, z.

3. Prove that $\dfrac{d}{dx}(\sinh^{-1}x)=(1+x^2)^{-\frac{1}{2}}$.

By expanding $(1+x^2)^{-\frac{1}{2}}$ in ascending powers of x and then integrating, show that when $|x|<1$

$$\sinh^{-1}x=\frac{x}{1}-\frac{1}{2}\frac{x^3}{3}+\frac{1.3}{2.4}\frac{x^5}{5}-\ \dots$$

and state the general term of the series. State without proof why the restriction $|x|<1$ is necessary.

Find the limit as x tends to zero of the function

$$\frac{x^2-x\sinh^{-1}x}{(\cosh x-\cos x)^2}.$$

4. (*a*) Show that $\displaystyle\int_0^\infty \frac{x\,dx}{(1+x^2)(4+x^2)}=\frac{1}{3}\log_e 2.$

(*b*) Find the maximum and minimum values of the function

$$\frac{x^2+1}{x^2+x+1}.$$

Sketch the graph of the curve whose equation is

$$y=\frac{x^2+1}{x^2+x+1}.$$

5. The radius of gyration of a uniform thin circular disc, of radius a, about a line through its centre and perpendicular to its plane it $a/\sqrt{2}$. Deduce that the radius of gyration of a uniform solid sphere of radius r about a diameter is $r\sqrt{(2/5)}$.

A spherical cavity of radius r is made inside a uniform solid sphere of radius R, in such a way that the two spherical surfaces touch at the point P. If G is the centre of mass of the remaining material, find (i) the distance PG, and (ii) the radius of gyration of the remaining material about a common tangent at P to the spherical surfaces.

6. A particle of mass m is projected vertically from the ground, its motion being subject to gravity and to an air resistance $R(v)$, a function of its speed v. When the particle is at a height x and moving upwards its speed v is given by

$$v^2 = ae^{-2gx/b} - b, \qquad (a > b > 0)$$

where a, b are constants and g is the constant acceleration due to gravity. Calculate, in terms of m, a and b, the work done against air resistance during the entire upward motion.

Find the acceleration as a function of v and deduce an expression for $R(v)$.

Show that in a downward motion of the particle starting from rest the speed can never exceed $\sqrt{b}$.

7. Two points A_1 and A_2, moving in a plane have coordinates (x_1, y_1) and (x_2, y_2) respectively, referred to axes fixed in the plane. State the components (parallel to the axes) of (i) the displacement $\overrightarrow{A_1A_2}$ and (ii) the velocity of A_2 relative to A_1.

At a certain instant a ship A_1 is sailing due east with speed U_1; a second ship A_2, north-east of A_1, is sailing due north with speed U_2; a third ship A_3, north-east of A_2, is sailing due south with speed U_3; also $A_1A_2 = A_2A_3 = d$. Assuming that the velocities of the three ships remain constant, find the easterly and northerly components of the velocities of A_2 and A_3 relative to A_1. Deduce expressions for the components of the displacements $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_1A_3}$ after time t.

Hence, or otherwise, show that the three ships are again in a straight line after a time

$$\frac{U_1+2U_2+U_3}{U_1(U_2+U_3)}\frac{d}{\sqrt{2}}.$$

8. A particle strikes a fixed smooth barrier at an angle α and rebounds at an angle β, both angles being measured from the perpendicular to the barrier at the point of impact. If the coefficient of restitution is e, show that $\cot\beta=e\cot\alpha$.

The interior of a smooth circular tray has the form of a hollow cylinder of small height, and is fixed with the base horizontal. A particle of mass m is projected horizontally with speed u from a point P on the circumference of the base, in a direction making an angle α with the diameter through P. If the particle returns to P after two rebounds, show that

$$\cot^2\alpha=e^{-1}+e^{-2}+e^{-3}.$$

9. A rough uniform rod, of mass m and length $4a$, is held on a horizontal table perpendicularly to an edge of the table, with a length $3a$ projecting horizontally over the edge. If the rod is released from rest and allowed to turn about the edge, show, by using the principle of energy, that its angular speed after turning through an angle θ is

$$\sqrt{\left(\frac{6g\sin\theta}{7a}\right)},$$

assuming that the rod has not started to slip.

Deduce an expression, in terms of θ, for the angular acceleration, and hence determine the reaction normal to the rod. Show that the rod begins to slip when $\tan\theta=4\mu/13$, where μ is the coefficient of friction.

(The moment of inertia of the rod about the edge of the table is $7ma^2/3$.)

June 1963

MATHEMATICS (ADVANCED)

PAPER I

(Three hours)

Negligently presented or slovenly work will be penalized.

Answer SEVEN questions.

1. (*a*) Given the arithmetic series

$$4+9+14+\ \dots,$$

obtain the sum of the first n terms. Hence find the least value of n for which the sum of the first n terms exceeds 2,000.

(*b*) Write down the sum of the series

$$a(1+r+r^2+\ \dots\ +r^7).$$

Hence or otherwise find the coefficient of x^5 in

$$x^7+x^6(1+x)+x^5(1+x)^2+\ \dots\ +(1+x)^7.$$

2. (*a*) Eliminate θ between the equations

$$k+\sin\theta = p\cos\theta,$$
$$k-\sin\theta = q\cos\theta,$$

where $p+q \neq 0$.

(*b*) Express $(5+4i)(3+2i)$ in the form $a+ib$, where a and b are real. Deduce a pair of factors of $7-22i$, and hence express 7^2+22^2 as a product of two positive integers.

(*c*) Express $(1+2i)^2$ and $(1+2i)^3$ in the form $a+ib$, where a and b are real. Hence find the real numbers r and s for which $1+2i$ is a root of the equation

$$z^3+rz^2-7z+s=0.$$

3. Sketch with the same pair of axes the graphs of the functions

$$y=(x-2)^2,$$
$$y=(x-2)^2+4,$$
$$y=(x-2)^2-4.$$

Indicate on each graph the coordinates of the turning point and of the points where the graph crosses the axes.

Find the range of values of k for which the equation

$$x^2 - 4x + k = 0$$

has unequal positive roots.

Write down the roots of the equation. Hence, given that k is so small that k^3 and higher powers of k may be neglected, use the binomial series to obtain approximations to the roots in the form of quadratics in k.

4. (*a*) Find the values of θ in the range $0 \leqslant \theta < 360°$ for which

$$\text{(i) } \sin\theta = \sin 28°,$$

$$\text{(ii) } \sin 2\theta + \sin^2\theta = 0.$$

(*b*) Given that

$$\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C,$$

show that

$$(\cos A + \cos B \cos C)^2 = \sin^2 B \sin^2 C.$$

Given also that A, B, C are positive acute angles, find $A+B+C$.

5. (*a*) Expand $\log_e \cos^2\theta$ in a series of ascending powers of $\sin^2\theta$, giving the terms up to the term in $\sin^6\theta$, and the general term. For what values of θ, in the interval $0 \leqslant \theta \leqslant \pi$, is the expansion valid?

(*b*) Given that

$$y = (2+x)^2 e^{-x},$$

find the expansion of y in ascending powers of x as far as the term in x^3.

Find also the expansion of $\log_e y$ in ascending powers of x as far as the term in x^3, and state the coefficient of x^n.

6. (*a*) Given the points $A(2, 14)$, $B(-6, 2)$, $C(12, -10)$, verify that the triangle ABC is right-angled. Calculate the coordinates of

(i) the point D on AB produced such that $AC = CD$,
(ii) the point of intersection of the perpendiculars from A, C, D to the opposite sides of the triangle ACD.

(*b*) The ends A, B of a rod of length c slide on two perpendicular lines Ox, Oy, the end A being on Ox. Show that the point P of the rod for which $AP:PB=1:2$ moves on an ellipse of centre O, and calculate the eccentricity of this ellipse.

7. Given that $v=u^3$ and $u=\cos t$, write down dv/dt in terms of t.

The parametric equations of a curve are

$$x=a\cos^3 t, \qquad y=a\sin^3 t.$$

Write down dx/dt and dy/dt and hence show that $dy/dx= -\tan t$. Deduce the equation of the tangent to the curve at the point t, and simplify the equation.

Show that the length of the perpendicular from the point $(a, 0)$ to this tangent is $| q |$, where

$$q=a(\sin t-\tfrac{1}{2}\sin 2t).$$

Calculate dq/dt, and hence find the maximum length of the perpendicular.

8. Given that $(x^2+1)y=x^2$, prove that for all values of x

(i) $y \geqslant 0$, (ii) $y \leqslant x^2$, (iii) $y<1$.

Show also that, if x is small, y is approximately equal to x^2.

Calculate the area bounded by the lines $x=0$, $x=1$, $y=1$, and the part of the graph of

$$y=\frac{x^2}{x^2+1}$$

between $x=0$ and $x=1$.

Show that the volume obtained by rotating this area through one revolution about the line $y=1$ is

$$\pi\int_0^1 \frac{dx}{(x^2+1)^2}.$$

By using the substitution $x=\tan\theta$, evaluate this integral.

9. (*a*) Given that $y = e^{ax} \sin bx$, where a and b are constants, prove that

$$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx}$$

is a constant multiple of y. *Deduce* that all positive stationary values of y are maximum values.

(*b*) Evaluate $\displaystyle\int_0^1 \frac{x^2 dx}{(1+x^3)^2}$.

(*c*) Evaluate $\displaystyle\int_5^6 \frac{dx}{(x-2)(x-4)}$,

giving the answer to three decimal places.

June 1963

MATHEMATICS (ADVANCED)

PAPER II

(Three Hours)

Negligently presented or slovenly work will be penalized.

Answer SEVEN questions.

1. Two uniform rods OA, OB, of lengths $3a$, $4a$ respectively and each of weight w per unit length, are smoothly jointed at O. The ends A and B are smoothly jointed to the ends of a *light* rod AB of length $5a$. The system is freely suspended from a fixed smooth pivot at O and hangs in equilibrium under gravity. Show that the inclination of OA to the downward vertical is $\tan^{-1}(16/9)$. Calculate the thrust in the light rod AB.

A couple of moment L is now applied to the system in the vertical plane OAB so that AB is horizontal in the new position of equilibrium. Find L in terms of w and a, and indicate the sense of the couple in a diagram.

2. A rigid body is in equilibrium under the action of three *non-parallel* coplanar forces. Show that the lines of action of these forces are concurrent.

To the end A of a uniform rod AB, of weight W and length $2a$, is attached one end of a light inextensible string of length l. To the other end of the string is attached a small light smooth ring Q which is free to slide on the rod. The system hangs in equilibrium under gravity with the string passing over a smooth fixed peg O, and the rod making an angle θ with the horizontal. Prove that

(i) OQ and AB are perpendicular,
(ii) the two portions of the string are each inclined to the vertical at the angle θ.

Calculate, in terms of W and θ, the tension in the string.
Show that

$$l \sin \theta = 2a \cos^3\theta.$$

3. Two equal uniform rough spheres, each of weight W, rest on a rough horizontal plane in contact with one another. A gradually increasing force P is applied to one of the spheres and acts along the line of centres towards the other sphere. Assuming that equilibrium is maintained, show that the magnitude of the frictional force is the same at each contact and equals $\frac{1}{2}P$. (The directions of these frictional forces must be clearly shown in a diagram.)

Determine the normal reactions at all three contacts and deduce that equilibrium is not possible if the coefficient of friction between the spheres is less than unity.

If μ is the coefficient of friction between each sphere and the plane, show that equilibrium is broken when P exceeds

$$\frac{2\mu W}{1+\mu}.$$

4. A river of width a ft. with straight parallel banks flows due north with speed u ft. per sec. The points O and A are on opposite banks and A is due east of O. Coordinate axes Ox, Oy are taken in the east and north directions respectively. A boat, whose speed V ft. per sec. relative to the water is constant, starts from O and crosses the river.

(i) If u is constant and equal to $V/6$ and the boat is steered so that it travels in a straight line towards A, find the time taken for the boat to travel from O to A.

(ii) If u varies in such a way that

$$u = x(a-x)V/a^2$$

and if the boat is steered due east, show that the co-ordinates (x, y) of the boat satisfy the differential equation

$$\frac{dy}{dx} = \frac{x(a-x)}{a^2}.$$

If the boat reaches the east bank at C, calculate the distance AC and find the time taken.

5. A particle P is released from rest on the smooth outer surface of a fixed sphere, of radius a and centre O, when the inclination of OP to the upward vertical OA is 60°. Show that while P remains in contact with the sphere,

$$a\left(\frac{d\theta}{dt}\right)^2 = g(1-2\cos\theta),$$

where θ is the angle POA at time t after release. Deduce that P leaves the sphere when $\theta = \cos^{-1}(\frac{1}{3})$ and find the vertical component of the velocity of P at that moment. Hence show that P reaches the level of O at the time

$$\frac{(\sqrt{13}-2)\sqrt{(2a)}}{3\sqrt{(3g)}}$$

after leaving the sphere.

6. A train of total mass 660 tons starts from rest and moves up a hill of inclination $\sin^{-1}(1/64)$. The engine works at a constant rate of 1056 horse-power and the frictional resistance to the motion of the train is 15 lb. wt. per ton. Show that the acceleration of the train when its speed is v ft. per sec. is

$$\frac{88-5v}{7v}\text{ft. per sec.}^2$$

Hence calculate, in miles per hour, the greatest possible speed of the train and show that the time during which the train acquires half this speed is

$$308\ (2\log 2-1)/25 \text{ sec.}$$

(Take g as 32 ft. per sec.^{2})

7. A light elastic string is of natural length l and modulus $4mg$. Show by integration that the work required to extend the string from length l to length $l+x$ is $2mgx^2/l$.

One end of the string is attached to a fixed point O of a smooth horizontal table and a particle P of mass m is attached to the other end of the string. The particle is released from rest on the table, with the string straight and $OP = 5l/4$. Find the velocity of P when $OP = l+x$, where $x>0$. Show that P reaches O at the time

$$(\tfrac{1}{4}\pi+2)\sqrt{\left(\frac{l}{g}\right)}$$

after release.

8. Two scale-pans, each of mass m, are connected by a light inelastic string which passes over a small smooth fixed light pulley. In one scale-pan there is an inelastic particle A of mass $2m$. The system is released from rest with the hanging parts of the string vertical. Find the tension in the string and the acceleration of either scale-pan. Find also the force exerted on A by the scale-pan.

At the instant when motion begins a particle of mass $3m$ is allowed to fall from rest and after t sec. it strikes and adheres to A. Find the impulsive tension in the string and the velocity of either scale-pan immediately after the impact.

9. A small smooth ring P of mass m is threaded on a fixed horizontal wire Ox. A second equal smooth ring Q is threaded on a fixed vertical wire Oy which points downwards from O. The rings are connected by a light inextensible string of length l and are released from rest with Q at O and $OP = l$. At time t after release the inclination of PQ to the horizontal is θ. Show that the downward velocity and acceleration of Q are

$$l\frac{d\theta}{dt}\cos\theta, \qquad l\left[\frac{d^2\theta}{dt^2}\cos\theta - \left(\frac{d\theta}{dt}\right)^2 \sin\theta\right]$$

respectively.

Show by energy considerations that

$$l\left(\frac{d\theta}{dt}\right)^2 = 2g\sin\theta$$

and deduce the value of $d^2\theta/dt^2$ in terms of l, g and θ.

Write down the equation of motion of Q and hence find, in terms of m, g and θ, the tension in PQ.

June 1963

FURTHER MATHEMATICS (ADVANCED)

PAPER I

(Three Hours)

Negligently presented or slovenly work will be penalized.

Answer SEVEN questions.

1. (*a*) Find the values of a for which the equations

$$3x-ay=5,$$
$$2ax+6y=-a,$$
$$5x+2ay=1$$

are consistent, giving the solution in each case.

(*b*) If $b \neq 1$, find all the values of x which satisfy the equation

$$\begin{vmatrix} x^3+1 & x^2 & x \\ b^3+1 & b^2 & b \\ 2 & 1 & 1 \end{vmatrix} = 0.$$

2. It is given that the equation $ax^4+4bx+c = 0$ has two equal roots. If λ denotes either of these roots, show that

$$a\lambda^3+b = 0$$

and deduce that

$$ac^3 = 27b^4.$$

Given that the equation

$$27x^4-32x+16 = 0$$

has two equal roots, find all its roots.

3. Prove that

$$1^2+2^2+3^2+\ldots+n^2=\frac{n(n+1)(2n+1)}{6}.$$

A non-degenerate triangle is to be made from three rods chosen from $2n$ straight rods whose lengths are 1, 2, 3, . . . , $2n$ units respectively. If the length of the longest side is $2r$ units, show that $(r-1)^2$ such triangles can be made.

Deduce that the total number of possible triangles whose longest side is an even number of units is

$$\frac{n(n-1)(2n-1)}{6}.$$

4. (*a*) For the curve $y \sinh x = 2$, prove that

$$1+\left(\frac{dy}{dx}\right)^2=\tfrac{1}{4}(y^2+2)^2.$$

Show that the radius of curvature at the point (x, y) on the curve $y \sinh x = 2$ is

$$\left|\frac{(y^2+2)^2}{4y}\right|.$$

(*b*) If ϕ is the angle between the tangent and the radius vector at the point (r, θ) on the curve $r = f(\theta)$, prove that

$$\tan\phi = r\frac{d\theta}{dr}.$$

If the equation of the curve is $r = 3+2\cos\theta$, find the values of ϕ when $\theta = 0$ and $\theta = \pi$, and sketch the curve. Also find the area enclosed by the curve.

5. Find the general solution of the differential equation

$$\frac{d^2x}{dt^2}+2\frac{dx}{dt}+2x = 0.$$

A particle moves in a plane in such a manner that its co-ordinates (x, y) satisfy the equations

$$\frac{dx}{dt} + 2x - y = 0, \qquad \frac{dy}{dt} + 2x = 0.$$

Prove that

$$\frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 2x = 0,$$

and hence determine x and y, given that $x = 1$, $y = 1$ when $t = 0$.

6. Forces $(P, 2P)$ and $(3P, P)$ act respectively at the origin O and the point $A(2a, a)$, the axes Ox, Oy being rectangular. Show that the resultant anticlockwise moment about the point (x, y) is

$$P(4y - 3x - a).$$

If the line of action of each of the four component forces is turned through an angle θ in the anticlockwise sense, show that the force at O now has components $P(\cos\theta - 2\sin\theta)$, $P(\sin\theta + 2\cos\theta)$ parallel to Ox, Oy respectively, and write down the components of the force at A. Show that the resultant force acts in the line

$$\cos\theta(4y - 3x - a) - \sin\theta(3y + 4x - 7a) = 0,$$

and that, whatever the value of θ, this line passes through a fixed point. Find the coordinates of the point.

7. A particle P, of mass m, can move in a fixed straight line AB, where $AB = a$, under forces of repulsion $k/(AP)^2$ from A and $4k/(BP)^2$ from B, where k is a positive constant. Find the position, O, of equilibrium.

If P is displaced from O towards B and then released, express the equation of motion in terms of the subsequent displacement x from O and d^2x/dt^2. Hence obtain the energy equation.

If x/a is always so small that $(x/a)^2$ is negligible, show that P moves with a simple harmonic motion of period

$$\frac{2\pi}{9}\sqrt{\left(\frac{ma^3}{k}\right)}.$$

8. A bullet of mass m is fired horizontally with speed V into a wooden block of mass M. While the bullet and block are in relative motion the resistance to penetration is of constant magnitude R. If the bullet penetrates to a depth a when the block is fixed, show that

$$R = \tfrac{1}{2}mV^2/a.$$

Instead of being fixed the block is now free to move (without rotation) along a smooth horizontal plane. Show that if, at time t, the displacement of the block is x and the depth of penetration is y,

$$m\frac{d^2}{dt^2}(x+y) = -R.$$

Explain why the total momentum of the bullet and block remains constant and write down an equation which expresses this fact. Deduce that the depth of penetration in this case is $Ma/(M+m)$.

(The effect of gravity is to be neglected.)

9. A uniform solid cube of edge a slides, with four of its edges horizontal, down an inclined plane. When moving with speed v the cube strikes a small inelastic horizontal ledge on the plane so that its lowest edge becomes fixed. Show that the angular speed of the cube just after the impact is $\frac{3}{4}\ v/a$, stating clearly the principle used in your solution. (Assume that the moment of inertia of the cube about an edge is $2ma^2/3$, where m is the mass.)

Find the loss of kinetic energy due to the impact. If the slope of the plane is β $(<\frac{1}{4}\pi)$, find the smallest value of v^2 which will cause the cube to topple over the ledge.

June 1963

FURTHER MATHEMATICS (ADVANCED)

PAPER II

(Three Hours)

Negligently presented or slovenly work will be penalized.

Answer SEVEN questions.

1. If a is not equal to 0 or ± 2, show that the curve

$$y = \frac{x^2 - 4}{(x+a)^2}$$

has one turning point and one point of inflexion, and state the corresponding values of x.

Find the Cartesian equation of the locus of the turning point as a varies.

2. (*a*) Use de Moivre's theorem to find the four fourth roots of $8(-1+i\sqrt{3})$ in the form $a+ib$, giving a and b correct to two decimal places.

(*b*) The complex numbers $(z-2)$ and $(z-2i)$ have arguments which (i) are equal, (ii) differ by $\frac{1}{2}\pi$, and each argument lies between $-\pi$ and $+\pi$. In each case find the locus of the point P which represents z in the Argand diagram, and illustrate by a sketch.

3. Prove that the equation of the plane which cuts off intercepts a, b, c on the axes of x, y, z respectively is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}= 1.$$

The foot of the perpendicular from the origin to a plane is $P(2,-1,2)$. Find the equation of the plane. If the plane meets the axes of x, y, z at A, B, C respectively, prove that AP is perpendicular to BC, and find the angle between AP and CP.

4. (*a*) Find

$$\int \frac{x^2 dx}{\sqrt{(x^2-1)}},$$

and hence find

$$\int x \cosh^{-1} x \, dx.$$

(*b*) Prove that

$$p \int \cos^n x \sin px \, dx =$$

$$-\cos^n x \cos px - n \int \cos^{n-1} x \sin x \cos px \, dx.$$

Deduce that

$$(p+n) \int \cos^n x \sin px dx =$$

$$-\cos^n x \cos px + n \int \cos^{n-1} x \sin[(p-1)x] \, dx.$$

5. A particle is projected from a point on a plane whose inclination to the horizontal is α. The velocity of projection is V at an angle θ to the horizontal and at $\theta - \alpha$ to the plane. Show that the range R on the plane is

$$2 \frac{V^2}{g} \cos\theta \sin(\theta - \alpha) \sec^2 \alpha.$$

If V and α are kept constant but θ is varied, find the maximum value of R.

A stone is thrown from a point P so as to hit a point Q whose horizontal and vertical distances from P are 108 ft.

and 45 ft. respectively. If the level of Q is above that of P, show that the speed of projection must be at least 72 ft./sec.

(Take g as 32 ft./sec.^{2})

6. Two particles of masses M and m ($M > m$) are connected by a light string which passes over a smooth fixed peg and hangs vertically on each side. The air resistance on each particle is kv^2 when the speed is v. Show that, if the resisted motion under gravity could continue indefinitely, the speed would tend to a limiting value V, where

$$V^2 = \frac{(M-m)g}{2k}.$$

Show that the acceleration of either particle, at speed v, has magnitude

$$\frac{(M-m)\ (V^2-v^2)}{(M+m)\ V^2}\ g.$$

Find the distance moved by either particle in attaining a speed of $\frac{1}{2}V$ from rest.

7. A small sphere falling vertically strikes with speed u a smooth plane inclined at an angle α ($<\frac{1}{4}\pi$) to the horizontal, and rebounds horizontally. The coefficient of restitution is e. Show that

(i) if the plane is fixed, $e = \tan^2\alpha$;

(ii) if the plane is kept moving horizontally (in the direction of the rebound) with constant speed V,

$$e = \frac{\tan^2\alpha - (V/u)\tan\alpha}{1+(V/u)\tan\alpha}.$$

8. A point moves in a circle of radius a with constant speed $a\omega$. Find the acceleration in magnitude and direction.

The axis of a rough hollow cylinder of internal radius a is horizontal and fixed. The cylinder is rotated about the axis with constant angular speed ω, and a particle on the inside curved surface is carried round with it, remaining at relative rest. Find the frictional force and the normal reaction on the particle when the radius to the particle makes an angle θ with the upward vertical.

If $\lambda = a\omega^2/g$, show that $\lambda > 1$ is the condition that the normal reaction is always positive. Assuming this to be the case, and that μ is the coefficient of friction between the particle and cylinder, show also that

$$\mu \geqslant 1/\sqrt{(\lambda^2 - 1)}$$

is the condition that the particle never slips.

9. Prove that the period of small oscillations of a rigid body moving under gravity about a smooth fixed horizontal axis is

$$2\pi\sqrt{\left(\frac{I}{Mgh}\right)},$$

where M is the mass of the body, I its moment of inertia about the axis and h the distance of its centre of mass from the axis.

Three equal uniform rods AB, BC, CD, each of mass m and length $2a$, are rigidly joined together at B and C so as to form three sides of a square, and the whole can rotate freely about a horizontal axis through A perpendicular to the plane of the rods. Find the moment of inertia of the system about the axis of rotation and the period of small oscillations about that axis.

(Proofs of formulae relating to moments of inertia are not required.)

June 1960

MATHEMATICS (SCHOLARSHIP)

(Three hours)

Answer SEVEN questions.

Candidates need not confine their attention to the questions which correspond to the Alternative they offer in Mathematics, Advanced, Paper II. For the information of candidates these questions have A, B or C respectively in front of the number.

1 sheet of graph paper supplied. Additional sheets will be supplied on request but ALL sheets issued must be placed within the answer-book and handed in to the Supervisor.

Special tables will be supplied for use in answering Question B 14.

1. The roots of the quadratic equation

$$x^2 - px + q = 0$$

are α and β. Form quadratic equations whose roots are

$$\text{(i) } \alpha + \frac{1}{\beta}, \quad \beta + \frac{1}{\alpha},$$

$$\text{(ii) } \alpha + \frac{1}{\alpha}, \quad \beta + \frac{1}{\beta},$$

and whose coefficients depend only on p and q.

Find in each case the conditions satisfied by p and q if the quadratic equation so obtained has equal roots.

2. (*a*) Find x if the sum of the infinite series

$$1 + \frac{1}{x} + \frac{1}{3}\left(x + \frac{1}{x^2}\right) + \frac{1}{3^2}\left(x^2 + \frac{1}{x^3}\right) + \frac{1}{3^3}\left(x^3 + \frac{1}{x^4}\right) + \dots$$

is $2/(x-1)$.

(*b*) The sides of a triangle are of lengths a, b, c. Show that

$$a^2+b^2+c^2 < 2(bc+ca+ab).$$

3. Sketch from $\theta=0$ to $\theta=4\pi$ the spiral curve given in polar coordinates by

$$r=ae^{m\theta}, \quad (a>0, m>0).$$

The lines $\theta=\beta$, $\theta=\gamma$ cut the curve in successive points B_1, B_2 and C_1, C_2 respectively, B_1 and C_1 being the first intersections outwards from the origin O. Find the ratio OB_1/OB_2, and show that B_1C_1 is parallel to B_2C_2.

If $\beta=\pi/2$, $\gamma=3\pi/4$ and C_1B_1 is parallel to the axis $\theta=0$, show that

$$m=\frac{1}{\pi}\log_e 4.$$

4. Find the equation of the tangent at the point $T(ct, c/t)$ to the hyperbola $xy=c^2$. Show that if the tangent at T touches the parabola $y^2=8(x-k)$ at P, then the ordinate of P is $-4t^2$.

Find the values of c and k if the tangents at the points of the hyperbola corresponding to $t=1$ and $t=-2$ both touch the parabola.

5. (*a*) If $y \log_e y=x$, find dy/dx in terms of y. Find also the value of x for which dy/dx is infinite.

(*b*) Find $\displaystyle\int\frac{(x^2-1)dx}{x^4+3x^2+1}$

by means of the substitution $u=x+\dfrac{1}{x}$.

A 6. A light bead is threaded on a piece of wire bent into the form of a circular arc whose extremities are A and B. When the bead is at a point P it is pulled by forces of fixed magnitudes F_1 and F_2 towards A and B respectively. Prove that for all positions of the bead on the wire, the resultant of

F_1 and F_2 passes through a fixed point C situated on the circle of which the arc is a part.

The wire is sufficiently rough for equilibrium to be possible for all positions of the bead on the wire. Show that if F is the frictional force and N the normal reaction between the bead and the wire, the value of F/N is greatest when the bead is either at A or at B.

A 7. Two planes meet in a horizontal line CD and are inclined at angles of 45° to the horizontal. A uniform rod AB of length $4a$ rests horizontally with its ends A and B on the planes. The rod is at right angles to CD, and C is vertically below the rod. Points H and K of the rod are distant a from A and B respectively. The contact at A is smooth, but at B the coefficient of friction is $\mu(<\frac{1}{3})$. When a force T acts through K parallel to BC towards the plane ACD, the rod is on the point of slipping downwards at B. When a force T_1 acts through H parallel to CB away from the plane ACD, the end B is about to slip upwards. Show that $T_1=3T$.

A 8. An express train 300 ft. long and a motor car are approaching a level crossing, the railway track and the road being at right angles. When the front of the train is 180 ft. from the crossing the car is 360 ft. away and travelling at its top speed of 130 ft. per sec. If the maximum retardation which the car's brakes can produce is 20 ft. per sec.^{2}, show that a crash cannot be avoided if the train maintains any steady speed between 65 and 120 ft. per sec. (Ignore the length of the car.)

A 9. Two particles are projected simultaneously from the same point O with the same speed U and hit the same target on the horizontal plane through O. If the angle between the initial directions of motion is 2β, show that

(i) the distance between the particles after time t is $2Ut \sin \beta$, so long as neither has reached the target,

(ii) the difference between their times of arrival at the target is $\frac{2\sqrt{2}U}{g}\sin\beta$,

(iii) the difference in the maximum heights attained is $\frac{U^2}{2g}\sin 2\beta$.

A 10. A particle of mass m is attached to the end B of a light spring AB of natural length l and modulus mn^2l. Initially the spring is unstretched and lies on a smooth horizontal table with the end A fixed. The particle is projected along the table in the direction AB with velocity U, where $U < nl$, and after travelling a distance b strikes a second particle C of mass $3m$. Find the velocity V of the particle of mass m immediately before collision, in terms of n, U and b. If the two particles stick together after collision, show that the amplitude a_1 of the subsequent oscillation is related to the amplitude a of the oscillation which would have taken place in the absence of C by the equation

$$4a_1^2 = a^2 + 3b^2.$$

B 11. If the probability of an event occurring at a single trial is p, prove that the probability of exactly r occurrences of the event in n independent trials is

$${}_nC_r p^r (1-p)^{n-r}.$$

Present records show that 10 per cent of 14-year-old children have had their tonsils removed. If a large number of 14-year-old children are examined in groups of thirty and a frequency distribution is drawn up showing the number in each group who have had their tonsils removed, estimate the mean, the variance and the mode of the distribution.

B 12. When A and B play chess the chance of either winning a game is always $\frac{1}{4}$ and the chance of the game being drawn is always $\frac{1}{2}$. Find the chance of A winning at least three games out of five.

If A and B play a match to be decided as soon as either has won two games, not necessarily consecutive games, find the chance of the match being finished in ten games or less.

B 13. If $\phi(x) = \frac{1}{\sqrt{(2\pi)}} e^{-\frac{1}{2}x^2}$, show that $x^n\phi(x) \longrightarrow 0$ when $x \longrightarrow \infty$.

Evaluate $\int_{-\infty}^{\infty} |x|\phi(x)\,dx$.

Also, assuming that $\int_{-\infty}^{\infty} \phi(x)\,dx = 1$, evaluate

$$\int_{-\infty}^{\infty} x^2\phi(x)\,dx.$$

Hence obtain, for the distribution $\phi(x)$, the ratio of the mean deviation to the standard deviation.

B 14. During a test on a steam turbine, two observers A and B, using separate manometers a and b respectively, took two sets of pressure readings. Both instruments were connected to the same orifice, and, at equal intervals of time, observations were made alternately by each observer. The following table summarizes the results:

	Observer A using manometer a	*Observer B using manometer b*
No. of readings	100	100
Mean of readings (in. of mercury)	14.06	14.32
Standard deviation (in. of mercury)	0.80	0.80

(i) If it could be assumed that each set of 100 readings is normally distributed and that each set could be regarded as a random sample of the variations in pressure, determine whether or not the difference between the means is significant at the 5 per cent level.

(ii) Assuming that the 100 readings made by observer A approximated very closely to a normal distribution, construct their frequency distribution using the following class intervals: *less than* 13.0, 13.0 —, 13.5 —, 14.0 —, 14.5 —, 15.0 —, 15.5 *and over.*

(iii) It was found that the readings taken by observer A agreed closely with those expected of a normal distribution but that those taken by B did not. Assuming that both observers were equally reliable, state any deductions that might be made from the results of the test. If the main object of the test is to register variations in the pressure of the turbine, suggest ways of improving the test when it is repeated.

B 15. The two equal sides of an isosceles triangle are each of unit length and the angle θ included between them is rectangularly distributed between 0 and $\pi/6$. Show that the area y of the triangle is distributed between 0 and $\frac{1}{4}$ with a probability distribution

$$p(y)\,dy = \frac{12}{\pi}(1-4y^2)^{-\frac{1}{2}}dy.$$

Sketch the probability curve and calculate the mean and variance of the area of the triangle.

C 16. (*a*) Prove that

(i) $$\begin{vmatrix} b+c & c+a & a+b \\ m+n & n+l & l+m \\ q+r & r+p & p+q \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ l & m & n \\ p & q & r \end{vmatrix},$$

(ii) $$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$$

(*b*) The equation
$$x^4+8x^3+gx^2+22x+8=0$$
has two roots whose product is 2. Find the constant g and all the roots.

C 17. (*a*) Two circles of radii a, b have their centres a distance c apart, $(c > a+b)$. Show that the distances of the centres from the radical axis are
$$\frac{c^2+a^2-b^2}{2c}, \quad \frac{c^2+b^2-a^2}{2c}$$
respectively.

(*b*) The line of centres of a system of coaxial circles cuts the radical axis at O, and L is a limiting point lying inside a given circle C of the system. A line through L cuts the radical axis at H and the circle C at points P, Q. If M, N are the feet of the perpendiculars from P, Q respectively to the radical axis, prove that $PM.QN=OL^2$.

C 18. The foci of an ellipse are S and S', and LL' is the chord of the ellipse passing through S and perpendicular to the major axis. The tangent at L cuts the minor axis at D and LS' cuts the minor axis at E. If O is the centre of the ellipse, prove that
$$DE=ES=\frac{SD^2}{2OD}.$$

C 19. (*a*) A rectangular box has sides of length a, b, c, where $a > b > c$. An insect sets out from one corner of the box to crawl to the most distant corner. Show that its shortest path is of length
$$(a^2+b^2+c^2+2bc)^{\frac{1}{2}}.$$

(*b*) Two planes intersect in a line LM. The first plane cuts a sphere in a circle, centre A, radius a; the second plane cuts the sphere in a circle, centre B, radius b. If P is any point on LM, prove that

$$PA^2 - PB^2 = a^2 - b^2.$$

C 20. (*a*) By considering ${}_nC_r$, or otherwise, show that the product of any r consecutive integers is divisible by $r!$ Deduce that if n is an integer not divisible by 3, then

$$n(n^2-1)(n^2-4) \equiv 0 \pmod{360}.$$

(*b*) If x is an integer, show that

$$x^2 \equiv 0,\ 1 \text{ or } 4 \pmod 5.$$

Deduce that if a, b, c are integers and

$$a^2 + b^2 = c^2,$$

then at least one of a, b, c is divisible by 5.

C 21. (*a*) Obtain the general solution of the differential equation

$$\frac{dy}{dx} + y \tan x = \sec x.$$

(*b*) Given that

$$\frac{dy}{dx} = \frac{y+9x}{y+x}$$

use the substitution $y = zx$ to obtain an equation expressing dz/dx in terms of z and x. Solve this equation and deduce that

$$(3x+y)(3x-y)^2 = C,$$

where C is an arbitrary constant.

C 22. (*a*) If $u^n = \displaystyle\int_0^{\pi/2} x^n \sin mx \, dx$

where m is of the form $4r+1$, r being an integer, show that for $n \geqslant 2$

$$u_n = \frac{n}{m^2}\left(\frac{\pi}{2}\right)^{n-1} - \frac{n(n-1)}{m^2} u_{n-2}.$$

(*b*) Evaluate $\displaystyle\int_0^1 \frac{2+e^x}{1+e^x} \, dx.$

MATHEMATICS (SCHOLARSHIP)

1961

(Three hours)

Answer SEVEN questions.

1. (*a*) Find the values of the real variable x for which
$$x^2 - |x| - 6 > 0.$$

(*b*) Prove that
$$x^4 + y^4 - x^3y - xy^3 > 0,$$
where x and y are real and different.

(*c*) Given that the complex number $a+ib$ is a root of the equation
$$x^3 + qx + r = 0,$$
where q and r are real, show that a is a root of the equation
$$8x^3 + 2qx - r = 0,$$
provided that $b \neq 0$.

2. A square plate of diagonal $4a$ is supported in a vertical plane with its two lower sides on two small pegs at a distance b apart in the same horizontal line. Show that when the diagonal which passes between the pegs is inclined at an angle θ to the vertical, the height h of the centre of the plate above the pegs is
$$2a \cos \theta - \tfrac{1}{2}b \cos 2\theta.$$

Determine the maximum and minimum values of h, and the positions of the plate for which they occur,

(i) when $b < a$,

(ii) when $a < b < a\sqrt{2}$.

3. (*a*) Prove that

$$\int_0^{\pi/2} \cos^m\theta \, d\theta = \frac{m-1}{m}\int_0^{\pi/2} \cos^{m-2}\theta \, d\theta,$$

where m is an integer greater than 1.

Hence, or otherwise, show that

$$I_n = \int_0^{\infty} \frac{dx}{(1+x^2)^n}$$

satisfies the reduction formula

$$I_{n+1} = \frac{2n-1}{2n} I_n,$$

where n is a positive integer, and evaluate I_4.

(*b*) Two rods AB and BC, each 2 ft. long, are hinged together at B. The ends A and C are made to move in a straight groove away from one another, each with a constant speed of 3 ft. per sec. Determine the velocity and acceleration of B, in magnitude and direction, at the instant when the rods are perpendicular to one another.

4. (*a*) A sphere of radius a is cut by a plane at a distance h from its centre. Prove that the volume of the smaller part is

$$\tfrac{1}{3}\pi(2a^3 - 3a^2h + h^3).$$

(*b*) A cylindrical hole is bored through a solid hemisphere of radius a by a drill of radius b ($< a/\sqrt{2}$). The axis of the hole passes through the centre of the base of the hemisphere and is inclined at 45° to the base. Prove that the total surface area of the drilled hemisphere is

$$\pi[a^2 - b^2\sqrt{2} + 2(a+b)(a^2-b^2)^{\frac{1}{2}}].$$

[The surface area of a sphere of radius a intercepted between two planes cutting (or touching) the sphere at a distance l apart is $2\pi al$; the area of an ellipse with semi-axes of lengths c and d is πcd.]

5. Find the equation of the locus of the point of intersection of two tangents to the parabola $x=at^2$, $y=2at$ which meet at a constant angle θ.

Show that the vertices of equilateral triangles circumscribed to the parabola lie on the curve

$$y^2=(3x+a)(x+3a).$$

6. A fly whose speed of walking is always u is initially at the centre O of a gramophone record of radius a which is rotating with a constant angular velocity ω ($<u/a$). If the fly then walks so as to move in a fixed compass direction OA, show that it reaches the edge of the record at time

$$t=\frac{1}{\omega}\sin^{-1}\frac{\omega a}{u}.$$

If the fly immediately turns round and walks back, always heading directly towards O, and reaches the groove of radius b at a point P, find the distance of P from OA measured along the groove.

7. A uniform rod AB, of weight W_1 and length $2a$, is freely hinged about its end A which is fixed. A light inextensible string of length $2b$ ($<2a$), on which is threaded a smooth ring P of weight W_2, joins B to a point C vertically above A. The system is in equilibrium with B on the same level as C. Show that P is at the mid-point of the string.

If the angles CAB and CPB are θ and 2ϕ respectively, show that

$$aW_2\cos\theta=b(W_1+W_2)\cos\phi.$$

Show also that the tension in the string is

$$\frac{b}{2}\left[\frac{W_1(W_1+2W_2)}{a^2-b^2}\right]^{\frac{1}{2}}.$$

8. A goods train consists of an engine of mass pm and n wagons, each of mass m, joined by inelastic couplings which are each initially slack by a distance a. The engine, exerting a constant force kmg, begins to move forward on a level track and jerks the wagons successively into motion. Assuming that

there is no resistance to the motion, determine the speed of the engine immediately before the first wagon moves, and show that the speed v_1 immediately afterwards satisfies the equation

$$[(p+1)v_1]^2=2kpga.$$

If v_r is the speed immediately after the rth wagon has been started, show that

$$[(p+r+1)v_{r+1}]^2-[(p+r)v_r]^2=2k(p+r)ga$$

and, by combining all such equations when r takes the values $1, 2, \ldots, (n-1)$ successively, obtain the value of v_n.

9. From a point O on a plane inclined at 45° to the horizontal a particle is projected upwards with a velocity V inclined at an angle α to the plane; the particle falls on the line of greatest slope drawn upwards from O. If the particle strikes the plane at right angles, show that $\cot\alpha=2$.

If the coefficient of restitution between the plane and the particle is $\frac{1}{2}$, show that the point of second impact is at a distance $\frac{3\sqrt{2}}{10}\cdot\frac{V^2}{g}$ from O.

1961

FURTHER MATHEMATICS (Scholarship)

(Three hours)

Answer SEVEN questions.

1. A circle C of radius a rolls without slipping round the outside of the circle $x^2+y^2=4a^2$, the two circles being coplanar. Show that the point P on the circumference of C which is initially at the point $(2a, 0)$ traces the locus given by the equations

$$x=a(3\cos\theta-\cos 3\theta),\quad y=a(3\sin\theta-\sin 3\theta).$$

Obtain approximations for x and y when θ is small, as far as terms in θ^3, and sketch the form of the locus in the neighbourhood of the point $(2a, 0)$.

Sketch also the whole locus, and show that its total length is $24a$.

2. Four complex numbers p, q, r, s are represented on the Argand diagram by the points P, Q, R, S respectively. If PQ is equal and perpendicular to RS, prove that $s-r=\pm i(q-p)$.

The points representing the complex numbers z_1, z_2, z_3, z_4 are at the vertices of a square, taken in anti-clockwise order round the square. Express z_3 and z_4 as linear functions of z_1 and z_2. Hence or otherwise show that

$$z_3^2+z_1^2-2z_2z_4=0, \qquad z_4^2+z_2^2-2z_1z_3=0$$

Show that the converse of this result is not true; that is, show that there exists a set of values of z_1, z_2, z_3, z_4 satisfying the above equations such that the corresponding points do not lie at the vertices of a square.

3. (*a*) Sketch the curve $y=\dfrac{1}{(5-x)^{1/3}}$, and evaluate the integrals

$$\int_{-3}^{5} y\,dx \text{ and } \int_{6}^{7} y\,dx.$$

(*b*) For each of the following integrals determine whether it exists and, if so, obtain its value:

$$\text{(i) } \int_0^\infty \frac{dx}{x^2(a^2+x^2)}, \qquad \text{(ii) } \int_0^\infty \frac{x^3dx}{(a^2+x^2)^4}.$$

4. A line L which passes through the origin O has direction-cosines l, m, n. Find the equation of the plane perpendicular to L which passes through the point $C(x_0, y_0, z_0)$. If L intersects the plane in the point N, find an expression for the length of ON, and show that if $CN=d$,

$$d^2=(mz_0-ny_0)^2+(nx_0-lz_0)^2+(ly_0-mx_0)^2.$$

A circular cylinder of radius d has the line L as its axis. Find an equation satisfied by the coordinates (x, y, z) of all points on the surface of the cylinder. Deduce that the surface intersects the plane $z=0$ in the curve whose equation is

$$x^2(1-l^2)+y^2(1-m^2)-2lmxy=d^2.$$

5. Two parallel planes at a distance h apart intersect a sphere of radius a. Show that the area of surface of the sphere lying between the planes is $2\pi ah$.

Two solid spheres, of radii 4 and 1 and with centres A and B respectively, are fixed so that the distance AB is c. A point source of light O lies between the spheres on the line AB. Find the total area of the spheres illuminated by the light as a function of x, where $x=AO$. Show that this area is a maximum when $x=8c/9$ provided that $c \geqslant 9$. Explain why this is no longer true when $c<9$, and show that if $c=6$, the greatest area illuminated is $32\pi/5$.

6. A particle P of mass m moves along a straight line OX under a force of magnitude km/x^2 directed towards O, where $x=OP$. Find the work done by the force as P moves from the point $A(x=2a)$ to the point $B(x=a)$. If P is released from rest at A, find its velocity when it reaches B.

Find also an expression for the velocity of P at any point in the interval $a \leqslant x \leqslant 2a$, and show that the time taken to reach B is

$$\left(\frac{\pi}{2}\right)\sqrt{\left(\frac{a^3}{k}\right)}.$$

7. A system of coplanar forces is reduced to a force acting at the origin of rectangular axes in the plane, and a couple G. If the components (X, Y) of the force are not both zero, show that the system can be reduced to a single resultant force, and find the equation of its line of action.

The moments of such a system about three non-collinear points (x_1, y_1), (x_2, y_2), (x_3, y_3) are G_1, G_2, G_3 respectively.

Show that on reduction to a force (X, Y) at the origin and a couple G the value of X is given by

$$X\Delta = \begin{vmatrix} x_1 & G_1 & 1 \\ x_2 & G_2 & 1 \\ x_3 & G_3 & 1 \end{vmatrix}, \text{ where } \Delta = \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix},$$

and obtain similar expressions for Y and G. Explain why it is necessary for the three points to be non-collinear.

8. Show that the radius of curvature of the ellipse $x=a \cos\theta$, $y=b\sin\theta$ at the point whose parameter is θ is

$$\frac{(a^2 \sin^2\theta + b^2 \cos^2\theta)^{3/2}}{ab}.$$

A smooth curve in the form of this ellipse is fixed with the x-axis horizontal and the y-axis directed vertically upwards. A particle of mass m placed on the curve at its highest point $(\theta=\frac{1}{2}\pi)$ is slightly disturbed so that it slides down the curve towards the positive direction of the x-axis. Find an expression for the reaction between the curve and the particle as a function of θ during the period of contact. Deduce that at the point where the particle leaves the curve θ satisfies the equation

$$(a^2 - b^2)\sin^3\theta + 3b^2 \sin\theta - 2b^2 = 0.$$

Verify that there is a value of θ between 0 and $\frac{1}{2}\pi$ which satisfies this equation.

9. A rigid body is freely mounted on a fixed axis L passing through its centre of mass and is acted on by a force F whose moment about L is N (not necessarily constant). If θ is the angle turned through by the body at time t, show that the rate at which F is doing work on the body is $N\, d\theta/dt$. By the use of energy considerations deduce that $I\, d^2\theta/dt^2 = N$, where I is the moment of inertia of the body about L.

Two coplanar pulleys, each of radius a, are freely mounted on parallel horizontal axes L_1, L_2 about which their moments of inertia are I_1, I_2 respectively. The axis L_1 is vertically above L_2 and distant h from it. A taut light inextensible endless string passes round both pulleys, the parts of the string not in contact with the pulleys being vertical. A particle P of mass M is attached to one of the vertical parts of the string, and the system is released from rest with P at the level of L_1. Find the time taken for P to reach the level of L_2, assuming that the string does not slip on the pulleys.

June 1962

MATHEMATICS (SCHOLARSHIP)

(Three hours)

Negligently presented or slovenly work will be penalized.

Answer SIX questions.

1. (*a*) Given the simultaneous equations

$$x+yz=y+zx=z+xy,$$
$$x^2+y^2+z^2=6,$$

show that $x=1$ or $y=z$, and hence solve the equations.

(*b*) Find necessary and sufficient conditions that must be satisfied by the real non-zero constants a, b, c in order that

$$ax^4+2bx^2+c$$

shall be positive for all real values of x.

2. From a point on the side line of a football field at a distance $2h$ from a corner flag the angle between the directions to the goal posts at the same end as the flag is α. Denoting the angle between the directions to the nearer post and the flag by θ, show that

$$\tan^2\theta+\tfrac{1}{2}\frac{d}{h}\tan\theta+1-\tfrac{1}{2}\frac{d}{h}\cot\alpha=0,$$

where d is the distance between the posts.

If, when the distance $2h$ is changed to h, the angle α changes to 2α, determine d in terms of h and $\tan\alpha$.

3. (*a*) Show that

$$\int_1^2 \frac{dx}{e^x - 1} = \log_e(1 + e^{-1}).$$

(*b*) Find the area of the segment of the parabola $y^2 = x$ cut off by the straight line $y = x - 2$.

4. (*a*) The tangents from the origin O to the circle

$$x^2 + y^2 - 4x - 12y + 15 = 0$$

touch it at A and B. Show that the equation of the circle OAB is

$$x^2 + y^2 - 2x - 6y = 0.$$

(*b*) Show that the tangents to the rectangular hyperbola $x = ct$, $y = c/t$ at the points with parameters t_1 and t_2 meet at the point

$$x = 2c\,\frac{t_1 t_2}{t_1 + t_2}, \quad y = \frac{2c}{t_1 + t_2}.$$

Three variable points on the hyperbola have a fixed centroid $(c, \frac{1}{3}c)$. Show that the vertices of the triangle formed by the tangents at the three points lie on a rectangular hyperbola, whose centre is the point $(3c, c)$.

5. Two uniform circular cylinders A and B, of radii $4a$ and a respectively but of equal weight W, rest in contact on a horizontal plane with their axes horizontal. Their central circular cross-sections are in the same plane p, and the coefficient of friction is the same at all three contacts. The cylinder A is pushed against the cylinder B by a horizontal force P acting in the plane p and through the centre of gravity of A. Indicate in a diagram the directions of the frictional forces and normal reactions on each cylinder, and show that, so long as equilibrium is maintained, the frictional force is the same at each contact. Show also that in equilibrium the normal reaction between the cylinders is equal to P, and determine the normal reactions at the other two contacts.

If the coefficient of friction is greater than $\frac{1}{2}$, determine how equilibrium will be broken as the force P is gradually increased.

6. A uniform cube, of edge a and mass $5m$, is at rest on a smooth horizontal surface when it is hit by a bullet of mass m travelling horizontally with velocity U. The bullet strikes a face of the cube perpendicularly at the centre and is then resisted by a force equal to $mU/(2a)$ times its velocity relative to the cube. Show that x, the depth of penetration at time t, satisfies the differential equation

$$\frac{d^2x}{dt^2} = -\frac{3U}{5a}\frac{dx}{dt}.$$

Determine the velocity of the cube at the instant when the bullet emerges.

[Assume that the bullet moves in a straight line through the cube.]

7. Two particles P and Q, of masses $3m$ and m respectively, are connected by a light inelastic string which passes over a smooth fixed peg. The system is released from rest with the particles hanging freely. After a time t_0 the particle P impinges inelastically on a fixed horizontal platform, while Q is still well below the peg. Show that P will next strike the platform after a further time $3t_0/2$. Determine the total duration of the motion before the system finally comes to rest.

8. A particle P of mass m is attached by light inextensible strings to fixed points Q and R on the same level. When the strings are taut, the angles at P and Q in the triangle PQR are $\frac{1}{2}\pi$ and θ respectively and P is at a distance a from the line QR. The particle is originally held vertically above the line QR with the strings taut and is then projected with a horizontal velocity $(5ga)^{\frac{1}{2}}$ perpendicular to the plane PQR. When P reaches its lowest point the string PR breaks. If P thenceforward describes a horizontal circle in the manner of a conical

pendulum, show that $\cot \theta = 3$, and find the breaking tension of the string PR.

[It may be assumed that neither string becomes slack before P reaches its lowest point.]

1962

FURTHER MATHEMATICS (SCHOLARSHIP)

(Three hours)

Negligently presented or slovenly work will be penalized

Answer SIX questions.

1. (*a*) Obtain the roots of the equation $2z^4 = 1 - i\sqrt{3}$ each in the form $re\theta$.

(*b*) If the points representing the complex numbers a, b, c are not collinear and if

$$\frac{(z-a)(b-c)}{(z-c)(b-a)}$$

is real, show that the point z lies on the circle through the points a, b, c.

2. If the equation of a curve is given in polar coordinates (r, θ) and ϕ is the angle between the radius vector and the tangent at a point of the curve, show that $\tan\phi = rd\theta/dr$.

A straight rod PQ of length $2a$ is constrained to move so that its mid-point R describes a circle of diameter a while the rod passes through a fixed point O on the circle. Choosing polar coordinates with the origin at O and the line $\theta = 0$ along the diameter through O, show that P and Q trace the curve $r = a(1 + \cos\theta)$. Sketch the complete curve and show that the area enclosed by it is $3\pi a^2/2$.

If S is the point of the circle diametrically opposite to R, show that SP is normal to the curve.

3. Obtain necessary and sufficient conditions that p and q must satisfy in order that the quadratic equation

$$x^2+2px+q=0$$

may have (i) distinct real roots, (ii) distinct positive roots, (iii) distinct real roots lying between -1 and 2.

Draw pairs of rectangular axes Op, Oq and indicate by shading in three diagrams the regions within which the point (p, q) must lie in order that these conditions may hold in the three cases respectively.

4. Two opposite faces of a cube of edge a are the squares $ABCD$ and $EFGH$, the edges perpendicular to these faces being AE, BF, CG, DH. Taking AB, AD and AE as axes of x, y and z respectively, show that if P is a point in the line BF such that $BP/BF=t$, then the equation of the plane through the points C, H, P is

$$x+ty+z=a(1+t).$$

Deduce that the coordinates of the foot N of the perpendicular from A on to this plane are $(\lambda, t\lambda, \lambda)$, where

$$\lambda=a(1+t)/(2+t^2).$$

If t is allowed to vary, so that the plane CHP rotates about HC, show that N moves on a circle in the plane $AFGD$, the centre of the circle being the mid-point of AK, where K is the mid-point of HC. Find the radius of the circle.

5. A rectangular tank has vertical sides of depth h and a horizontal base of unit area. An inlet supplying water at a constant rate fills the tank in time T when running alone. An outlet, through which water flows at a rate proportional to the square root of the depth of water in the tank, empties the tank in time $4T$ when running alone. Show that, if x is the depth of water at time t when both outlet and inlet are running, then

$$\frac{dx}{dt}=p-q\sqrt{x},$$

where $p=\dfrac{h}{T}$ and $q=\dfrac{\sqrt{h}}{2T}$.

Deduce that if both inlet and outlet are running the initially empty tank will be filled in time

$$4T(2 \log_e 2 - 1).$$

6. A particle moves on a smooth curve under the action of gravity and the reaction of the curve only. Prove that the square of its velocity is proportional to its depth below a certain fixed level. Explain why the reaction of the curve does not enter into this result.

A particle of mass m is attached to one end of a light rod of length a. The other end of the rod is attached to a small bead of mass m threaded on a rough fixed horizontal wire. The system is released from rest with the particle in contact with the wire. Find the velocity of the particle when the rod is inclined at angle θ to the horizontal, assuming that slipping has not taken place. Find also the horizontal and vertical components of the force exerted on the bead by the wire, and show that if the coefficient of friction exceeds $\frac{3}{4}$ the bead will never slip.

7. Deduce from Newton's laws that if two smooth spheres collide obliquely the vector sum of their momenta is unaltered by the impact.

Two equal smooth spheres, P and Q, each of mass m, move on a rectangular billiard table $ABCD$, where $AB=2a$, $BC=2b$. The spheres are moving, each with speed u, on lines parallel to AB in opposite senses and collide at the centre of the rectangle, P being nearer to CD and moving in the sense A to B. The line of centres at impact is inclined at angle

$$\beta \left(0 < \beta < \frac{\pi}{2}\right) \text{ to } AB$$

and the coefficient of restitution between the spheres is e. Find the magnitude and direction of the vector representing the momentum transferred from Q to P at the impact. If after the impact P moves in a straight line to the pocket C, show that $\tan\beta$ is the positive root of the equation

$$b\tan^2\beta - a(1+e)\tan\beta - be = 0.$$

(Neglect the size of the spheres compared with that of the table.)

8. Define *moment of inertia* and derive an expression for the kinetic energy of a body rotating with angular velocity ω about a fixed axis about which its moment of inertia is I.

A uniform circular disc, of mass m and radius a, can turn freely about a fixed horizontal axis through its centre and perpendicular to its plane. When the disc is at rest a particle of mass m moving with speed v in a horizontal line L strikes the rim of the disc and adheres to it. The line L is in the plane of the disc at a depth $a\sin\beta$ $(0 < \beta \leqslant \frac{1}{2}\pi)$ below the axis. Find the angular velocity of the system just after the impact. Find also the radial and tangential components of the impulse on the particle. If the disc subsequently makes complete revolutions, find the least value that v may have.

If the depth of L below the axis may be varied, find the least possible value that v may have for complete revolutions of the disc.

June 1963

MATHEMATICS (SPECIAL PAPER)

(Three Hours)

Negligently presented or slovenly work will be penalized.

Answer SIX questions.

1. (*a*) Prove that the circle $x^2+y^2=6$ lies entirely inside the circle $x^2+y^2-2x-4y = 17$.

(*b*) If the circles

$$x^2+y^2 = a^2, \qquad (x-c)^2+y^2 = b^2$$

have no real common point, prove that

$$(a+b+c)(b+c-a)(c+a-b)(a+b-c) < 0.$$

2. Given that $a^2 \neq 1$, express as the sum of three partial fractions the function $f(x)$ given by

$$f(x) = \frac{1}{(1-x)(1-ax)(1-a^2x)}.$$

Show that the coefficient of x^n in the expansion of $f(x)$ in ascending powers of x is

$$\frac{(1-a^{n+1})(1-a^{n+2})}{(1-a)(1-a^2)}.$$

Find also the expansion of $f(x)$ in ascending powers of x when (i) $a = 1$, (ii) $a = -1$, stating the coefficient of x^n in each case.

3. (*a*) Prove that

$$\sin^3 (\theta+30^\circ)+\sin^3 (\theta-30^\circ) \equiv \tfrac{3}{4}\sqrt{3} \sin \theta.$$

(*b*) A square is inscribed in a regular hexagon, two sides of the square being parallel to two sides of the hexagon. Prove that the ratio of the area of the square to that of the hexagon is $4(2-\sqrt{3})/\sqrt{3}$.

4. (*a*) Sketch the graph of $y = 1+2 \cos x$ from $x = 0$ to $x = \pi$. Evaluate the integral

$$\int_0^\pi |1+2 \cos x| \, dx.$$

(*b*) Find the value of λ for which the integral

$$\int_0^1 (\lambda x+\sin \pi x)^2 \, dx$$

has its minimum value, and find the minimum value.

5. A uniform ladder *AB*, of weight *W*, rests in equilibrium with the end *A* on the ground and the end *B* leaning against a vertical wall. The ladder makes an angle of 45° with the ground and with the wall, and the vertical plane through the ladder is perpendicular to both. The normal and frictional components of the reaction at *A* are N_1, F_1 and those at *B* are N_2, F_2, respectively. Show that

$$N_1=\tfrac{1}{2}W+F_1, \qquad N_2=F_1, \qquad F_2=\tfrac{1}{2}W-F_1.$$

If the coefficient of friction at *A* and at *B* is $\frac{1}{2}$, show that

$$\frac{W}{3} \leqslant F_1 = N_2 \leqslant \frac{W}{2}, \quad 0 \leqslant F_2 \leqslant \frac{W}{6}, \quad \frac{5W}{6} \leqslant N_1 \leqslant W.$$

6. (*a*) A certain substance decays at a rate which at any instant is proportional to the amount of the substance remaining at that instant. If x_0 is the amount at time t_0 and x_1 is the amount at time t_0+t_1, show that the amount at time t_0+nt_1 will be x_1^n/x_0^{n-1}.

(*b*) A particle is projected with velocity u at time $t = 0$ and moves in a straight line. At time t the velocity is v and the distance travelled is x. The motion is opposed by a resistance kv^n per unit mass, where k is a positive constant. Show that the equation of motion is

$$dv/dt = -kv^n.$$

Prove that, if $n<1$, the particle will come to rest when

$$t = \frac{u^{1-n}}{(1-n)k} \text{ and } x = \frac{u^{2-n}}{(2-n)k}.$$

If $n = 1$, show that the particle will never quite come to rest, and that x will never quite reach a certain limiting value.

If $n = 2$, show that the particle will never come to rest and that x will increase without limit.

7. Three elastic spheres A, B, C, equal in all respects, lie at rest on a smooth horizontal table, with their centres in a straight line, in the order A, B, C. Sphere A is projected directly towards B with velocity u. If the coefficient of restitution is e, show that after three collisions the velocities of the spheres will be respectively

$$(1-e)(3+e^2)u/8, \qquad (1-e^2)(3-e)u/8, \qquad (1+e)^2u/4.$$

Show that there will be no further collision if $e \geqslant 3-2\sqrt{2}$.

8. Sketch the graph of the function $(1-\cos\theta)/\theta$ in the interval from $\theta = 0$ to $\theta = 2\pi$. Show that the maximum value of the function in this interval is $\sin\alpha$, where α is the smallest positive root of the equation $\theta = \tan\frac{1}{2}\theta$.

A simple pendulum consists of a bob of mass m suspended from a frictionless pivot by a light rod of length l. When the pendulum is hanging in equilibrium a force of constant magnitude F begins, and then continues, to act on the bob in a direction always perpendicular to the rod and always in the same vertical plane.

Obtain the energy equation in the form

$$\tfrac{1}{2}ml(d\theta/dt)^2 = F\theta - mg(1-\cos\theta),$$

where θ denotes the angle through which the rod has turned in time t.

Show that the rod will make complete revolutions if F exceeds $mg\sin\alpha$, where α is the angle referred to in the first part of the question.

June 1963

FURTHER MATHEMATICS (SPECIAL PAPER)

(Three Hours)

Negligently presented or slovenly work will be penalized.

Answer SIX questions

1. (*a*) The complex numbers z_1, z_2, z_3 are represented by the points A, B, C respectively in the Argand diagram. If

$$2z_1^2 + z_2^2 + z_3^2 - 2z_3z_1 - 2z_1z_2 = 0$$

show that AB and AC are equal and perpendicular to each other.

(*b*) State and prove de Moivre's theorem in the case when the index n is a positive integer.

Show that the roots of the equation

$$x^8 - x^7 + x^6 - x^5 + x^4 - x^3 + x^2 - x + 1 = 0$$

are $\cos\frac{r\pi}{9} + i\ sin\frac{r\pi}{9}$ where $r = 1, 3, 5, 7, 11, 13, 15, 17.$

Deduce that

$$\cos\frac{\pi}{9} + \cos\frac{3\pi}{9} + \cos\frac{5\pi}{9} + \cos\frac{7\pi}{9} = \frac{1}{2}.$$

2. Find the number of ways in which n identical coins can be placed (i) in two different boxes, (ii) in three different boxes, all the coins being used and empty boxes being allowed in

each case. Show that the number of ways in which the coins can be placed in four different boxes is

$$\frac{1}{2}\sum_{s=0}^{n} (s+1)(s+2).$$

If ${}_nF_r$ denotes the number of ways in which the n coins can be placed in r different boxes, show, by considering the cases when one particular box is empty or has at least one coin, or otherwise, that

$${}_nF_r = {}_{n-1}F_r + {}_nF_{r-1}.$$

Use this relation to construct a table of values of ${}_nF_r$ for $n =$ 1, 2, 3, 4, 5 and $r =$ 1, 2, 3, 4, 5.

3. If s is the length of arc of a curve $y = f(x)$ measured from an arbitrary point on it, show that

$$(ds/dx)^2 = 1 + (dy/dx)^2.$$

In a certain curve which passes through the origin the length of arc satisfies the relation

$$s = a \log_e \frac{a}{a-x}, \ (a > 0).$$

Show that the curve must lie between the lines $x = 0$ and $x = a$ and that it can be represented parametrically by the equations

$$x = a(1-\cos t),$$
$$y = a\log_e(\sec t + \tan t) - a\sin t.$$

Sketch the complete curve in the range $-\frac{1}{2}\pi \leqslant t \leqslant \frac{1}{2}\pi$.

If the curve makes a complete revolution about the line $x = a$, show that the area of the surface generated is $4\pi a^2$.

4. A line of gradient k passes through the point $A(0, a)$ and cuts the fixed line $y = c$, where $c > 0$, in the point P. Find

the coordinates of P. The line through A of gradient $1/k$ cuts the line OP in the point Q, O being the origin. Show that for all values of k the point Q lies on the curve whose equation is

$$cx^2+(a-c)y^2-a(a-c)y = 0.$$

Describe how this curve changes its character as the point A takes successive positions on the positive part of the y-axis, stating the name of the curve for all values of $a > 0$, and for $a = 0$. Show in particular that when $a = 5c$ the curve is an ellipse with semi-axes $5c$, $5c/2$.

5. A body A of mass M is connected by a light inextensible string to a vessel B containing liquid, the mass of the vessel and liquid being M at time $t = 0$. The string passes over a fixed smooth peg below which A and B hang and the system is initially at rest. The liquid evaporates in such a way that its mass decreases at a constant rate μ per unit time. Find the acceleration of the system at time t, and deduce that its speed is then

$\sin 2\theta = 2\sin\theta\cos\theta.$

$$-gt-\frac{2Mg}{\mu}\log_e\frac{2M-\mu t}{2M}.$$

$\cos 2\theta = \cos^2\theta - \sin^2\theta = 2\cos^2\theta - 1 = 1-2\sin^2\theta.$

Obtain an approximation to this expression when $\mu t/M$ is small, neglecting powers of t higher than the second. Deduce that the distance through which A has fallen at time t is at^3 approximately, where a is a constant, and find the value of a in terms of M, μ and g. If the mass which evaporates in one day is $\frac{1}{2}M$, find the approximate distance through which A has fallen at time $t = 60$ sec., taking the value of g to be 32 ft./sec.^{2}

6. A particle P of mass m moves in the x, y-plane under the action of forces $\mathbf{R}$ and $\mathbf{S}$ which are respectively tangential and normal to its path. These forces have magnitudes mkw and $m\lambda w$ respectively, where w is the speed of P, and k, λ are constants. The tangential force $\mathbf{R}$ opposes the motion and $\mathbf{S}$ is directed so that rotation through a right angle from $\mathbf{R}$ to $\mathbf{S}$

is in the same sense as that from the x-axis to the y-axis. Show that, if u, v are the x- and y-components of the velocity of P at time t, the equations of motion are

$$du/dt = \lambda v - ku, \quad dv/dt = -kv - \lambda u.$$

Deduce that

$$\frac{d^2u}{dt^2} + 2k\frac{du}{dt} + (\lambda^2 + k^2)u = 0.$$

If $u = 0$, $v = v_0$ when $t = 0$, show that at time t

$$u = v_0 e^{-kt} \sin \lambda t, \qquad v = v_0 e^{-kt} \cos \lambda t.$$

7. Two particles move in a plane under the action only of the mutual force between them. Prove that the sum of the components of their momenta in any direction is constant.

Two atomic nuclei of masses $3m$ and $2m$ move along perpendicular lines AO, BO with speeds u, $2u$ respectively and collide at O. On collision they are transformed into two nuclei of masses m and $4m$ which move with speeds v_1, v_2 respectively along a line XOX', where OX lies in the right angle AOB. Show that $\tan AOX = 4/3$. If at the collision the total kinetic energy is decreased by $\frac{1}{2}mu^2$, find the possible values of v_1, v_2.

Show also that in all cases in which the transformed nuclei move along XOX' after the collision, the loss of energy at the collision cannot exceed $3mu^2$.

8. The angular momentum h about the origin of a particle P of mass m moving in a plane is given by

$$h = m\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right),$$

where x, y are the coordinates of P at time t. Deduce that the moment about the origin of the resultant force acting on P is equal to dh/dt.

A uniform rod of mass M and length $2a$ is freely pivoted at its centre to a fixed point, and is initially at rest in a horizontal position. A particle of mass m, released from rest at a height c above the rod, strikes it at a distance x from the pivot and adheres to it. Find the angular velocity of the rod immediately after the impact. Show that the rod will subsequently make complete revolutions if

$$c > \frac{3mx^2 + Ma^2}{3mx}. \qquad (1)$$

Show that this condition cannot be satisfied for any value of x unless $c > 2a\sqrt{(M/3m)}$.

If $M > 3m$, show that the condition (1) cannot be satisfied unless $c > (3m+M)a/3m$.

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